Standards for the Preparation of Teachers of Mathematics

Middle Grades (grades 5-9) and High School (grades 7-12)

Michigan State Board of Education

Approved

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Participants in Standards Development

Jennifer Bell, Special Education Resource Room Teacher, Troy School District

Kathy Berry, President, Michigan Council of Teachers of Mathematics (MCTM); Director of Research, Evaluation, and Assessment, Monroe County Intermediate School District (MCISD)

Dr. Esther Billings, Professor, Department of Mathematics, Grand Valley State University

Gail Burrill, Mathematics Specialist in the Program in Mathematics Education, Michigan State University; Past President of the National Council of Teacher of Mathematics

Dr. Stephanie Casey, Associate Professor, Eastern Michigan University

Jennifer Curtis, Mathematics Education Consultant, Oakland Schools

Marcus Deja, Mathematics Specialist, Teaching and Learning Department, Kent Intermediate School District

Dr. Thomas G. Edwards, Professor of Mathematics Education, Wayne State University

Jason Gauthier, Mathematics Education Consultant, Allegan Area Educational Service Agency

Dr. John Golden, Associate Professor, Grand Valley State University

Anne Harkema Penn, Testing Coordinator and Mathematics Teacher, Michigan Great Lakes Virtual Academy

Dr. Jon Hasenbank, Associate Professor of Math Education, Department of

Mathematics, Grand Valley State University

Dr. Vicki-Lynn Holmes, Associate Professor of Mathematics and Education, Hope College

Deb Hubbard, Research Assistant, University of Michigan

Tracy Krafft, High School Math Teacher, Godwin Heights Public Schools

Dr. Angela S. Krebs, Associate Professor, Department of Mathematics and Statistics, University of Michigan – Dearborn

Dr. Douglas A. Lapp, Professor of Mathematics & Mathematics Education, Central Michigan University

Dr. Jennifer M. Lewis, Associate Professor of Mathematics Education, Wayne State University

Dr. Nancy K. Mack, Professor, Department of Mathematics, Grand Valley State University

Dr. Eric Mann, Associate Professor of Mathematics, Hope College

Bridgett McDowell, Education Assessment Specialist, Michigan Department of Education

Dr. Laura McLeman, Associate Professor of Mathematics, University of Michigan – Flint

Darcy McMahon, Higher Education Consultant, Michigan Department of Education

Stephanie Nimene, Assistant Superintendent of Curriculum, Instruction and Assessment, Crescent Academy

Eric Retan, Teacher, Wyoming Public Schools

Marie Smerigan, Mathematics Education Consultant, Oakland Schools

Dr. Mary Starr, Executive Director, Michigan Mathematics and Science Leadership Network

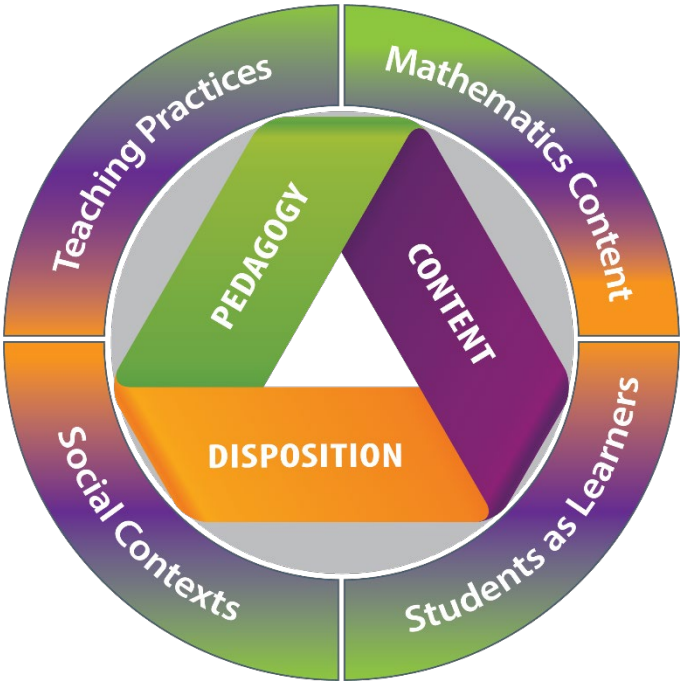
Luke Wilcox, Teacher, Kentwood Public Schools

Introduction

These standards for mathematics teacher preparation represent several shifts from current standards in that they: embed Core Teaching Practices throughout; are equity focused; are written as performance objectives for beginning teachers of mathematics; integrate pedagogy, content, and dispositions; and have a narrower but deeper focus on content through conceptual understanding and application particularly needed by teachers. The four domains of the standards were intentionally ordered to reflect these shifts: A. Pedagogical Knowledge and Practices for Teaching Mathematics; B. Students as Learners of Mathematics; C. Understanding of Social Contexts of Mathematics Teaching and Learning; and D. Understanding Mathematics Concepts, Practices, Dispositions, and Curriculum. Educator preparation programs are expected to implement these standards in tandem with the Professional Knowledge and Skills standards and with Clinical Experiences Requirements in order for teacher candidates to experience coherent, connected preparation experiences across the grade band(s) in which they are seeking endorsement.

Theoretical Foundation

Mathematics teacher preparation addresses development of teachers' professional proficiencies in three areas: pedagogy, content, and dispositions (PCD). Pedagogy refers to the ways of teaching mathematics, content refers to the what of teaching mathematics, and dispositions refers to the beliefs and attitudes about mathematics, mathematics teaching, and mathematics learners.



Too often the preparation of teachers has taken a siloed approach to developing teacher proficiency in these areas, addressing each in isolation from the others. The authors of these standards advocate for an approach that integrates the development of these proficiencies throughout the teacher preparation program both within and across courses and experiences. Such integration provides opportunities to prepare beginning mathematics teachers for the reality of the work of teaching where they will be continuously called to integrate their proficiency across all three areas. Therefore, each standard across the four domains (Teaching Practices, Students as Learners, Social Contexts, and Mathematics Content) will be supported by performance objectives that describe proficiencies concerning pedagogy, content, and disposition for that standard.

Foundational Principles

Two principles should undergird all teacher preparation programs in secondary mathematics. First, all aspects of a teacher program should attend to equity. Equitable teacher preparation programs provide opportunity, support, and advocacy for every potential teacher candidate and ensure that candidates are prepared to enact these three components of equity for every student in their care and for all students impacted by the system. This principle is explicitly called out in Domains A (Pedagogical knowledge and practices for teaching mathematics), B (Students as learners of mathematics) and C (Understanding of social contexts of mathematics teaching and learning) but may not be as visible in Domain D (Understanding mathematical concepts, practices, dispositions and curriculum). Attention to equity is still important in relation to content, however, given the structural barriers often in place that restrict access to rigorous mathematics and reduce student agency (NCTM, 2018). These include beliefs, structures, and practices that work to exclude some students from having equitable mathematical opportunities because they are subject to decisions made about their ability, temperament, and future by the system.

The second principle concerns the use of technology in the teaching and learning of mathematics and is related to equity. "Mathematical action technologies" (Dick & Hollebrands, 2011) can provide access to more and deeper mathematics for more students. They allow students to engage in mathematics that would have been out of reach without the technology, enable a focus on developing conceptual understanding and serve as platforms for exploration that includes making and testing conjectures. This means that planning for effective instruction (Domain A) that promotes student engagement (Domain B) should attend to the role that technology can play in student learning. Technology is also increasingly being used for virtual teaching and learning. Teacher preparation programs should ensure that teacher candidates are aware that many students have insufficient and inadequate access to technologies for virtual learning and learn how to provide accommodations to support learning in such situations.

The content preparation (Domain D) of beginning mathematics teachers should include experiences with tools and technological resources in all content domains,

including opportunities to use mathematical action technologies to explore mathematical relationships and deepen their mathematical understanding, to interpret mathematical representations, and to employ complex manipulations necessary to solve problems. These technological resources should not be restricted to advanced computer software packages which the candidates are unlikely to see or use in their work in schools but should include interactive applets, handheld graphing technology, dynamic geometry software and computer algebra systems that are appropriate for use at the secondary level. Mastery of skills should not be a prerequisite for using technology in any content area; rather, the focus when using technology should be on developing understanding and interpreting the results (Roschelle et al., 2000; Sacristán et al., 2010).

Beginning mathematics teachers should not view the increasing prevalence of software packages that can answer mathematical questions complete with steps and supporting rationales as obstacles. Rather, such tools should be viewed as opportunities for access to mathematics and whose existence highlights the need for teacher preparation programs to emphasize conceptualization, justification, and making sense of mathematical relationships in their work with prospective teachers.

Looking forward, the preparation of mathematics teacher candidates should recognize that what is important in mathematics is changing as the world and the affordances of technology change. The growing emphasis on big data or data science should be reflected in the mathematics and statistics teacher candidates will be expected to teach in secondary schools. In addition, coding and computational thinking are becoming increasingly important in all disciplines, and teaching aspects of computer science have typically been assigned to mathematics teachers. Teacher preparation programs should be aware of these shifts in what teachers will be expected to do and design experiences within their program to ensure their graduates are ready to meet the content demands they are likely to face in the field.

Grade Band Differentiation

These standards were developed by a single secondary mathematics stakeholder committee who attended to both the Middle Grades (grades 5-9) and High School (grades 7-12) grade bands. Purposefully, there is a significant amount of overlap and repetition between the grade bands. The first reason for this is that grades 7-9 are shared between the grade bands. Secondly, the focus on pedagogy and disposition throughout the standards is grounded in common elements. The differentiation between the grade bands is evident and concentrated in Standard D1 where the content knowledge needed for teaching each grade band is detailed. Further comparisons of content knowledge for teaching at the different grade bands can be found in Appendix C: Mathematics Grade Band Comparisons. In the remainder of the document, the differences between the grade bands are more subtle but nonetheless distinct and real. These domains and standards should be read and implemented with a deep understanding of the context that middle level and high school teachers will find themselves in, and, most importantly, with the understanding that they will need

to meet the learning needs of students in the different grade bands. While it has not been deemed necessary to specify in the standards themselves, any reference to "all learners" or "all students" should be understood to refer to all students in that grade band.

Notes on Content

Subject Matter Knowledge for Teaching

Research on the relationship between teachers' mathematical knowledge and students' achievement supports the importance of teachers' content knowledge in student learning. (Ball et al., 2008). Such content knowledge allows teachers to organize and use their knowledge effectively and to be able to respond appropriately to students during instruction in ways that further the students' learning (Hattie, 2011). According to a summary of several research projects by Walshaw (2012), teachers' decisions about instruction are shaped by their knowledge of the content to be taught. From a pedagogical perspective, without a clear understanding of mathematical or statistical ideas, teachers may resort to examples that lead to confusion, give inappropriate or unhelpful feedback, or misinterpret student solutions. "In short, teachers' fragile subject knowledge often puts boundaries around the ways in which they might develop students' understandings." (Walshaw, 2012, np)

In addition to knowing the mathematics their students will learn, well-prepared beginning teachers need specialized mathematical knowledge essential for teaching. They need an understanding of the concepts behind the mathematics, as well as the mathematical work they will do as a teacher, for example, helping students understand the role of structure in solving equations (e.g., $3(2x - 5) = 18$) rather than simply following a set procedure. These are demanding activities that require knowledge different from simply knowing the mathematics students are learning. The knowledge of content teachers will use in the work of teaching includes:

- Common content knowledge - the knowledge of mathematics described in the Michigan PK-12 Mathematics Standards that teachers will be expected to teach at their grade band
- Knowledge at the mathematical horizon - knowledge of mathematics above and below their grade band including how mathematical concepts are connected, how ideas develop and progress across grades (e.g., how the definition of a fraction as a number on the number line relates to adding rational expressions in algebra or building from a proportional relationship in grade 7 to linear equations in algebra)
- Specialized content knowledge - mathematical knowledge and skills unique to the work of teaching (e.g., choosing examples that deliberately confront misconceptions such as "solving" an expression or using technology to investigate the relationship between $\sqrt{a^2 + b^2}$ and $a + b$).

One implication of this is that teacher candidates at the middle grades level need to complete study of concepts from advanced algebra, trigonometry and introductory calculus in order to be prepared to teach grades 5-9 mathematics. It is expected that middle grades teachers will be well-prepared to teach mathematical and statistical concepts through first-year algebra and geometry. A teacher candidate prepared to teach high school should understand how key concepts at the high school play out in

later mathematics. High school teachers should be well prepared to teach typical courses such as those described in Catalyzing Change in High School Mathematics (2018) particularly those courses that address the Michigan K-12 Mathematics Standards. The terms basic or foundational and robust or comprehensive have been used to clarify and describe the level of understanding required for teaching middle grades (5-9) and high school (7-12). Note that this does not mean that middle grades teachers' understanding is always basic and high school teachers' understanding is always robust. A combination of robust and basic understanding of specific essential concepts are required for middle level candidates to develop specialized content knowledge and knowledge at the mathematical horizon to teach at the middle grades and likewise at the high school level.

Pedagogy of Content Classes

Many secondary mathematics teacher candidates will have experienced success with a narrow school mathematics curriculum that did not promote conceptual knowledge or emphasize mathematical practices and processes. Thus, in order to be able to teach in ways that develop their students' mathematical understanding, prospective teacher candidates should gain personal experiences with those practices and the ways they can support deeper knowledge of important mathematical concepts. This means the course work prospective teachers encounter in their mathematical preparation, including calculus, statistics and advanced mathematics courses, must be taught in ways that are consistent with what we know are effective teaching practices. Secondary school mathematics teachers may major in mathematics or have a strong mathematical focus, but the theoretical mathematics and statistics courses that are typically offered by many universities are often taught predominantly through lecture (Freeman et al, 2014). This does not sufficiently prepare secondary mathematics teachers. Secondary mathematics teacher candidates must have opportunities in their own mathematical learning to critique the reasoning of others, explain and defend their thinking and make and test conjectures. The mathematics courses they take should engage them in developing conceptual as well as procedural knowledge, utilize tasks that have high levels of cognitive demand with multiple solution paths and focus on reasoning and sense making activities.

Cross Cutting Content

Two aspects of mathematical content—reasoning and proof, and mathematical modeling—cut across the content areas. Reasoning and proof constitutes a central part of mathematics. It involves exploring mathematical ideas as well as making, rejecting, and/or refining conjectures. To establish whether and why a conjecture does or does not hold involves reasoning about what is known, often from general theorems to specific instances, and can lead to the creation of arguments that might become proofs. The idea that logical conclusions can be established by using reasoning and proof and that changing assumptions or definitions can lead to

different conclusions is a way of knowing that is special to mathematics. Unlike other fields, where new knowledge may undermine old knowledge, statements in mathematics are not easily overturned by new knowledge. A geometry statement or trigonometric identity can be established, forever and for all cases, by reasoning deductively from definitions and assumptions. Statistical reasoning, however, is typically inductive and the reliability of a statistical claim typically has a quantifiable level of uncertainty. Secondary mathematics teacher candidates should have experiences with various forms of proof within and across the mathematics and statistics courses they take and be prepared to guide learners from informal reasoning to a mathematical proof or to quantify the likelihood of a statistical conclusion. Programs should emphasize reasoning, argumentation and proof in all content areas, not just geometry.

As the world becomes increasingly data driven and technology continues to open new doors to ways of thinking about the world, mathematical modeling is one aspect of mathematics that should be receiving more attention in the mathematical preparation of teachers. A mathematical model is a mathematical representation of a particular real-world process or phenomenon that is under examination, in an attempt to describe, explore, or understand it (NCTM, 2018). Modeling involves determining which aspects of the phenomenon to include in the model and which to ignore and what kind of mathematical representation to use. The mathematical modeling cycle begins with a real problem and involves stating assumptions, formulating the problem mathematically, using a mathematical representation to solve the problem, analyzing the solution and if necessary refining the model. (NGA Center and CCSSO 2010a; Consortium for Mathematics and Its Applications and Society for Industrial and Applied Mathematics [COMAP and SIAM] 2016). Modeling provides an avenue for understanding and critiquing the world in which we live, and teacher candidates should have opportunities to engage with modeling activities within and across each of the content domains to investigate how different phenomena might behave or different events might unfold under given constraints or assumptions. These experiences can provide the background for the kinds of activities teacher candidates might implement in their own classrooms.

The content standards in this document are intended to highlight key concepts within each domain and are not intended to be exhaustive. They were developed by considering the following within each content area: learning progressions, engagement and appreciation for mathematics and of doing mathematics, connections within and across content domains, applications to mathematical and real world situations, procedural and conceptual knowledge, language and notation, and common underlying structures within a domain as well as attention to aspects of reasoning and proof, modeling and technology that are particularly relevant for a given domain.

A. Pedagogical Knowledge and Practices for Teaching Mathematics

A.1. Promote Equitable Teaching

Well-prepared beginning teachers of mathematics structure learning opportunities and use teaching practices to advance the learning of every student by providing access, support, and challenge while learning rigorous mathematics. Well-prepared beginning teachers of mathematics:

a. Facilitate a range of tasks through equity-based pedagogy including consideration of students' individual needs, cultural experiences, and interests, as well as prior mathematical knowledge.

b. Develop a classroom community in which students present ideas; challenge one another's ideas respectfully; construct meaning together; value and celebrate varied mathematical strengths; and use mathematics to address problems and issues in their school, homes, and communities.

c. Ensure all student approaches, responses, representations, experiences, and voices are valued in mathematical inquiries, discourse, and problem solving.

d. Facilitate multiple opportunities for all students to formulate, represent, analyze, and interpret mathematical models using a variety of tools including technology.

e. Provide all students access to the ways of doing mathematics (e.g., creating chains of reasoning and logic based on definitions and theorems, using simulations to investigate mathematical situations, and making and testing conjectures) and opportunities to communicate their thinking using appropriate mathematical language and representations.[1]

f. Engage all students in challenging mathematics content, building from their own funds of knowledge as they use multiple representations and models of their choice.

[1] Corrected from the original: A.1.e: Provide all students access to the ways of doing mathematics (e.g., inquiry, technology, mathematical language including symbols and notation).

g. Use students' developing understandings as found in various student representations (e.g., visualizations, vocalizations, models, symbols, and notations) to appropriately plan next steps for instruction.

A.2. Plan for Effective Instruction

Well-prepared beginning teachers of mathematics attend to a multitude of factors in planning for effective instruction (e.g., learning progressions, students' individual learning needs, options for student engagement, task selection and implementation, and formative and summative assessment data). Well-prepared beginning teachers of mathematics:

a. Establish appropriate and rigorous learning goals for students, which build on student understandings and are situated within learning progressions, research about student learning, mathematics standards and practices, and the approach to learning mathematics.

b. Attend to the development of both conceptual and procedural understanding as they choose tasks and design instruction.

c. Plan and implement rich tasks, including the appropriate instructional strategies that provide opportunities and access for all students to actively engage in the mathematical learning.

d. Anticipate an array of students' responses to tasks, craft questions, and prepare follow-up replies to probe student thinking in a way that relates the mathematical concepts and procedures.

e. Select mathematics-specific tools and technology to develop student conceptual understanding of mathematics.

f. Plan ways to use evidence of student thinking to assess progress toward mathematical understanding and possible instructional adjustments.

g. Draw on current research to develop mathematics instruction and assessment.

h. Consider their students as learners, including how to motivate and engage all students in learning mathematics.

A.3. Implement Effective Instruction

Well-prepared beginning teachers of mathematics implement effective mathematical pedagogical practices found in NCTM's *Principles to Actions* (2014) to develop

students' meaningful learning of mathematics. Well-prepared beginning teachers of mathematics:

a. Use established learning goals to guide instructional decisions.

b. Engage students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

c. Engage students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.

d. Facilitate discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student strategies and arguments.

e. Pose purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

f. Use strategies that ground procedural fluency in conceptual understanding so that students, over time, become skillful in using procedures flexibly and efficiently as they solve contextual and mathematical problems.

g. Provide students with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

h. Use evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

i. Analyze students' thinking that leads to an incorrect answer, identify the mathematical understanding that students have and may not yet have, and reply in a way that helps students develop their own understanding.

j. Implement the use of appropriate mathematical tools (e.g., technology and manipulatives) to develop students' conceptual understanding.

k. Are committed to enacting effective mathematics instruction that supports the learning of each and every student, including appreciating and accepting student reasoning that may be atypical or different from their own.

l. Reflect upon instruction to inform planning for future teaching.

A.4. Enhance Teaching Through Collaboration With Colleagues, Families, and Community

Well-prepared beginning teachers of mathematics seek and leverage collaboration with others, which could include education professionals, parents, caregivers, and community partners, to provide effective mathematics instruction for every student. They also engage colleagues, families, and/or the community in mathematical conversations and experiences to enhance the mathematical learning of their students. Well-prepared beginning teachers of mathematics:

a. Make their professional ideas and decisions visible and subject to shared examination by colleagues to develop a deeper shared understanding of students' learning and to support their development as a teacher.

b. Plan strategies to grow professionally and promote the mathematical success of students in collaboration with colleagues.

c. Understand, interact, and build intentional relationships with families and community members to learn their perspectives, priorities, and cultures that can provide information regarding the mathematical learning needs of students.

d. Utilize mathematics knowledge and experiences related to students' culture, community, family, and history as resources for interactions with students.

e. Provide constructive feedback to students' families and caregivers focused on strengths and areas of growth concerning students' mathematical performance.

f. Develop shared strategies with families and caregivers for promoting mathematical success of the students.

g. Possess an awareness of and commitment to employing multiple strategies to get to know students' families and communities in order to better serve students.

B. Students as Learners of Mathematics

B.1. Anticipate and Attend to Students' Thinking About Mathematics Content

Well-prepared beginning teachers of mathematics understand how students' mathematical ideas develop across levels of thinking in learning a specific mathematical topic. They use knowledge of what comes before and after a given mathematics topic to plan instructional moves to extend student thinking and competency. Well-prepared beginning teachers of mathematics:

a. Understand learning progressions within the mathematical content domains they may teach which includes knowledge of the mathematics that comes before and after a given mathematics topic.

b. Elicit and analyze students' thinking to understand where students lie on the learning progression.

c. Utilize understanding of students' thinking to plan for and execute instructional moves to advance students' learning.

d. Provides frequent opportunities for students to be metacognitive about their own learning and understandings.

e. Recognize the importance of eliciting and understanding student experiences and identity in shaping their mathematical thinking.

B.2. Promote Students' Engagement in Mathematical Practices

Well-prepared beginning teachers of mathematics strategically create environments in which students engage in the mathematical practices as described in the Michigan K-12 Mathematical Standards and recognize the challenges that students may face when engaging in them. They can identify teaching moves that support or inhibit students' engagement with the mathematical practices. Well-prepared beginning teachers of mathematics:

a. Anticipate that students will present various approaches to problems and support students in analyzing, comparing, justifying, and proving their solutions.

b. Create environments in which strategies are respectfully discussed, critiqued, and compared.

c. Identify how contexts, culture, conditions, and language can be used to create meaningful and authentic tasks that relate to their students' interests, backgrounds, prior knowledge, and experience, leading to increased engagement and motivation in math.

d. Present tasks that have high cognitive demand, have multiple solution strategies, entry points and representations, require communication of thinking and reasoning, and ensure students engage in rigorous mathematics.

e. Reflect on teaching moves that support or inhibit the engagement of students with the mathematical practices.

f. Pose questions for students that help them analyze problem situations, select strategies, and reason quantitatively.

B.3. Cultivate Positive Mathematical Identities of Students

Well-prepared beginning teachers of mathematics support students in developing and maintaining positive mathematical identities. A student with a positive mathematical identity has a productive disposition towards mathematics that includes the inclination to see mathematics as sensible, useful, and worthwhile, along with a belief in one's own diligence and efficacy when doing mathematics. Well-prepared beginning teachers of mathematics:

a. Are reflective practitioners, understanding that student learning of mathematics relies upon the teacher and should view their roles as supporting the development of students' mathematical identities through their interactions with students and instructional decisions.

b. Plan and implement mathematics instruction that draws on all students' mathematical strengths and positive mathematical identities that will allow them to be successful with the mathematics they are learning which in turn continues to develop positive mathematical identities.

c. Analyze their task selection and implementation, reflecting on ways this shapes students' mathematical identities and consider how the experience of doing the task supports developing a positive mathematical identity for each student.

d. Create classroom environments and orchestrate classroom discussions that enable respectful communication about mathematical ideas that support the development of positive student identities.

e. Work to combat negative beliefs including stereotypes about who is good at math and build positive beliefs within and among students.

f. Exhibit an asset-based perspective rather than a deficit-based view in interactions with students, realizing that mathematical errors are opportunities for learning and that all students bring their own unique mathematical strengths to the learning environment.

g. Acknowledge the diversity of all individual and group identities, particularly those whose learning experiences and needs are different from their own, including both individual differences (e.g., personality, interests, ways of thinking and knowing, life experiences)[2], and group differences (e.g., race, ethnicity, ability, gender identity, gender expression, sexual orientation, nationality, language, religion, political affiliation, socio-economic background) and use these in designing instruction to validate and build productive identities.

h. Take conscious and intentional actions to build students' agency as mathematical learners, based on strong beliefs that each and every student can learn mathematics with understanding.

C. Understanding of Social Contexts of Mathematics Teaching and Learning

C.1. Provide Access and Opportunity

Well-prepared beginning teachers of mathematics work to provide access for every student to a high-quality mathematics curriculum, effective teaching and learning where high expectations, and the support and resources needed to maximize the learning potential of each and every student are in place. They are also knowledgeable about and prepared to advocate for equitable practices for identifying opportunities for all students to succeed and excel in mathematics. Well prepared beginning teachers of mathematics:

a. Understand that access includes ensuring all students, particularly those that have been historically marginalized, have qualified teachers, opportunities for placement into higher level courses, high quality curriculum, and opportunities to approach problems.

b. Understand how denial of access and advancement perpetuate and produce inequities in the learning of mathematics for all students, particularly for traditionally underrepresented and/or underserved students.

[2] Corrected from the original: (e.g., personality, interests, learning modalities, life experiences)

c. Are knowledgeable about and prepared to advocate for equitable practices for identifying students for advancement, which include analysis of multiple indicators ensuring advancement is open to a wide range of students.

d. Understand the negative impacts of tracking students into qualitatively different or dead-end course pathways.

e. Understand and can employ accommodations available for students with exceptionalities to promote their access and advancement in mathematics.

C.2. Understand Power and Privilege in the History of Mathematics Education

Well-prepared beginning teachers of mathematics understand the social and economic impacts of the roles of power and privilege in the history of mathematics education and are equipped to promote systems that produce equitable mathematics learning experiences and outcomes for all students. Well-prepared beginning teachers of mathematics:

a. Understand current and historical mathematical educational practices that contribute to inequitable student opportunities and outcomes such as, but not limited to: social barriers (e.g., finances, teaching philosophy, demographic trends, culture), structural barriers (e.g., classroom size, schools, teachers, resources), and system policies (e.g., those related to placement and instruction, tracking, high stakes test taking).

b. Are knowledgeable about national reform movements in mathematics education, including the strides and challenges in affording every student a quality mathematics education.

c. Recognize and are willing to advocate for changes to policies and procedures that have negatively impacted mathematics learning, particularly for those students who have not historically experienced success in mathematics.

d. Analyze mathematical curriculum and instruction to determine whether either is likely to contribute to inequitable mathematical outcomes and opportunities for students.

e. Recognize implicit and explicit biases in themselves and others, including biases in the school/district culture, which work against equitable mathematics learning opportunities and support for all learners; works to counter these biases so that all students learn challenging mathematics deeply and well.

C.3. Enact Ethical Practice for Advocacy

Well-prepared beginning teachers of mathematics are prepared to take action in the classroom, at school meetings, in professional settings, with families and the wider community, and in other spaces to publicly champion meaningful and robust mathematical experiences for all students. They hold themselves and others accountable to challenge the status quo of their students' mathematical experiences and recognize the advocacy role of teaching. Well-prepared beginning teachers of mathematics:

a. Develop and use language that is effective in advocating for all students and conveys high expectations for learning mathematics.

b. Identify personal beliefs, classroom practices, and systemic structures that produce equitable and inequitable mathematical learning experiences and outcomes for students.

c. Work with others to develop strategies and methods to ensure traditionally marginalized students experience success in mathematics.

d. Are knowledgeable about and use effective advocacy strategies that promote meaningful inclusion of all students in the learning of mathematics.

e. Recognize their responsibility to stand up to exclusion, prejudice, and injustice of students in the learning of meaningful and robust mathematics.

D. Understanding Mathematics Concepts, Practices, Dispositions, and Curriculum

NOTE: Bold Text = required for 7-12 grade band, *Italics* = required only for 5-9 grade band, standard text = required for both grade bands. See also Appendix C for lists separated by grade band.

D.1. Know Relevant Mathematical Content

Well-prepared beginning teachers of mathematics have deep and flexible knowledge of core mathematical concepts and procedures that they will teach, along with knowledge both beyond what they will teach and foundational to those core concepts and procedures.

D.1.1. Essential Concepts in Number

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts in number including flexibly applying computational procedures, using real and rational numbers in contexts, solving problems involving numbers and evaluating the solutions. Essential concepts in number include *integers, fractions*, number theory, operations (including exponentiation) and their meanings and properties, meanings of complex numbers**, operations with complex numbers, matrices, vectors, and graphing in a polar coordinate system.** Well Prepared beginning teachers of mathematics:

a. Demonstrate "number sense"- flexible reasoning with and about whole numbers, integers and rational numbers in a variety of situations and applications through opportunities such as composing and decomposing numbers and number talks.

b. Describe the underlying structure of the *real*/**complex** number system and the learning progression for the development of number across the grades from kindergarten through high school.

c. Identify and apply a variety of strategies to compare and estimate rational and irrational numbers.

d. Understand and are fluent in using operations and appropriate notation, including exponentiation **and logarithms**, with *rational*/**complex** numbers, and can apply and justify multiple strategies for adding, subtracting, multiplying, and dividing *rational*/**complex** numbers.

e. Apply and connect concepts such as factor, prime, divisible, and multiple to particular numbers and sets of numbers.

f. Reason about and prove basic theorems about real numbers (e.g., the product of two negative numbers is positive, $\sqrt{2}$ is irrational, or the product of two odd numbers will be odd).

g. Use technology to investigate certain numbers (e.g., value of pi, compare the relative size of two numerical expressions, evaluate limiting processes, or **compute with matrices**).

h. Understand how complex numbers are related to the solutions of *quadratic*/**polynomial** equations.

i. **Recognize that vectors and polar coordinates are useful mathematical tools to describe both location and direction and are able to employ these tools in contextual situations.**

j. **Recognize that matrices are a shorthand notation for organizing information and for carrying out computations on that information.**

D.1.2. Essential Concepts in Ratios and Proportional Relationships

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts related to ratios and proportional relationships including rates and percents and their applications in contexts. Essential concepts in ratios and proportional relationships include understanding ratios; that equivalent ratios connect to proportional relationships (y=kx) and to graphs of rays from the origin[3] (0,0); and that percents are a particular form of a ratio and are invaluable tools in comparing quantities when the group sizes are different. Well-prepared beginning teachers of mathematics:

a. Describe the learning progression for the development of proportional reasoning across the grades from kindergarten through high school.

b. Identify situations in which ratios can be a tool to solve problems and apply a variety of strategies such as ratio tables, double number lines, and unit rates to solve problems involving ratios.

c. Understand ratios as paired quantities that vary together in the same relationship, distinct from a single number, recognizing that a ratio a:b may be associated with a value a/b (if b is not 0) and can describe the differences and similarities between ratios and fractions.

d. Identify and use equivalent ratios, reason about the role of multiplication and addition in generating equivalent ratios and recognize that the sum of equivalent ratios is another equivalent ratio.

e. Associate a unit rate with a ratio, recognize that equivalent ratios have the same unit rate, and recognize this unit rate as the constant of proportionality (k) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

f. Connect the constant of proportionality to the slope of a line[4] through the origin and to the equations of such a line $a:b \rightarrow y = (b/a)x$, and describe how this perspective relates to the general concept of linearity.

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[3] clarified from the original language: "graphs of rays from the origin"

[4] changed for clarity from the original: "proportionality (k) to the slope of a line"

g. Reason about contextual situations, identifying those that involve proportional relationships and use a variety of strategies for solving problems involving proportions, including finding a unit rate.

h. Connect ratios to scale factors, both within and between scaled figures, and relate scale factors to similar shapes, including how scaling a figure affects the area and volume of the scaled figures.

i. Describe percentage as a particular ratio of a quantity to 100 and apply this understanding to solve a variety of contextual problems.

D.1.3. Essential Concepts in Algebra

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major concepts in algebra including how algebra can be used systematically to represent patterns and relationships among numbers and other objects, to analyze change, and to model everyday events and problems of life and society. *Well-prepared beginning middle school (5-9) teachers are expected to have a comprehensive, robust understanding of the essential concepts related to algebra content at the 5-9 level*; the **well-prepared beginning high school (7-12) teachers will have a comprehensive, robust understanding of algebra and a basic understanding through linear algebra**. Essential concepts in algebra include understanding the role of and generating equivalent expressions, solving equations and inequalities in both real **and complex numbers**, and graphing algebraic relationships. **Additional concepts should include algebra from a more theoretical approach such as relationships among groups, rings, and fields and concepts from linear algebra.** Well-prepared beginning teachers of mathematics:

a. Explain how algebra as the language of generalization is useful for describing patterns and relationships.

b. Appreciate the value that new technologies can bring to developing algebraic understandings and strategically employ them in improving learning experiences in algebra for all students.

c. Describe how algebraic concepts build from arithmetic and are connected to other content areas, such as geometry, statistics, and calculus.

d. Describe the role of and be able to apply definitions, reasoning and proof in algebra including identifying conditions under which theorems are valid, recognizing contradiction as a proof strategy, and using conjectures to investigate algebraic relationships.

e. Use different technologies to enhance the learning of mathematics such as computer algebra systems to investigate algebraic structures and to check results; spreadsheets to produce and explore regularity in repeated reasoning; graphs to explore algebraic relationships; and interactive dynamic technologies to develop conceptual understanding of key algebraic topics.

f. Interpret the structure of an algebraic expression in terms of a context and understand that structure can provide insight into a mathematical situation.

g. Connect symbolic, graphical, tabular, and verbal representations of a problem or situation and explain any advantages and disadvantages of each representation for the given problem or situation.

h. Use algebra as a tool to solve contextual problems including identifying variables, formulating an algebraic model, manipulating and analyzing the model, interpreting the results, and validating the conclusions.

i. Explain and justify routine procedures involved in manipulating expressions and solving equations including the use of the properties related to multiplication, addition, and equality.

D.1.4 Essential Concepts in Functions

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge, and applications of the major concepts related to function, including models of real-world contexts where one variable determines another. The differentiation between a well-prepared middle grades and high school teacher lies in their additional level of expertise in the various types of functions noted below. Essential concepts in function include the notion of functions as a mapping[5], the characteristics of different function families (e.g., linear, exponential, polynomial, absolute value, piece-wise defined, **trigonometric**, **logarithmic**) and creating and interpreting different representations of functions. Well-prepared beginning teachers of mathematics:

a. Recognize the value of function as the language and organizational structure in the analysis of mathematical relationships.

b. Understand how algebra concepts are related to the ideas of function and that not all algebraic equations are functions.

[5] Reworded from original to ensure accuracy, as defining the term was unnecessary: "Essential concepts in function include mapping from one set to another in which each element of the second set is uniquely paired with one element from the first, recognizing . . ."

c. Represent functions, with and without technology, in a variety of ways including mapping diagrams, function notation, recursive definitions, tables, and graphs.

d. Connect members of the same function family and identify distinguishing attributes (structure) common to all functions within that family.

e. Compare function families and describe their similarities and differences (e.g., linear functions are additive, exponential functions are multiplicative).

f. Describe and reason about key features of the graphs of functions, using appropriate language, (e.g. zeros, intercepts, rate of change, increasing/decreasing and maximum/minimum values, and **asymptotes**); associate symbolic representations with these features and interpret them in both mathematical and real-world contexts.

g. Model a wide variety of real situations using functions and understand the processes of making and changing assumptions, assigning variables, and finding solutions to contextual problems.

h. Apply one or more function transformations to a representation (verbal, symbolic, graphical or tabular) of a function and explain the effects and results on other representations.

D.1.5 Essential Concepts in Statistics and Probability

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major concepts in statistics and probability including understanding the statistical investigation process (formulate questions, collect data or select portions of given data sets, analyze data, interpret results). Essential concepts in statistics and probability include aspects of data literacy, visualizing and summarizing data while recognizing the foundational role of variability, statistical inference, and basic ideas related to probability. Well-prepared beginning teachers of mathematics:

a. Describe the difference between the way conclusions are made in mathematics (deductive and deterministic) and in statistics (inductive and account for uncertainty) and appreciate that statistical reasoning is always grounded in a context in the presence of variability.

b. Describe statistics as an investigative process of problem-solving and decision making and explain how proficiency in statistical thinking matures as learners gain more knowledge and experience with variability (the developmental levels as outlined in the Guidelines for Assessment and Instruction in Statistics Education).

c. Use real data with a context and purpose, hands on activities and active learning and technology to explore concepts and to manage and analyze data in developing understanding of statistical ideas.

d. Understand and explain measures of center and variability[6] and relate them to distributions of data.

e. Identify the role of randomization and chance in determining the probability of events and **how randomization is connected to the scope of inference**.

f. Draw inferences about a population based on a sample in light of sampling variability using simulation-based techniques that lead to *informal*/**formal** inference procedures.

g. Evaluate reports based on data, reasoning critically and asking questions about the implementation of the statistical investigation process and critique the ways in which numbers and graphical representations are used in the media, mathematical contexts, and everyday discussions.

h. **Describe the differences among types of studies (survey, experiment, and observational) and number and type of variables (quantitative and categorical) determine the scope and validity of statistical inferences.**

i. Connect the probability of an outcome to a long-run relative frequency of an outcome and investigate chance processes, developing, using, and evaluating probability models.

j. Analyze and describe the association between variables **including experience with multivariate thinking** and considering whether causation can be established.

k. Design a basic observational or **experimental** study based on the statistical investigation process.

l. Reflect upon data from implementation of classroom practices to enhance their own craft.

D.1.6 Essential Concepts in Geometry, Trigonometry, and Measurement

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major concepts in geometry, recognizing geometry as a systematic study of size, shape,

[6] updated from the original "Understand and explain measures of center and spread" for clarity

location and navigation to provide a lens to view and model the world. Essential concepts in geometry, trigonometry, and measurement include properties of one, two, and three-dimensional shapes; measurement techniques and units; transformations; geometric arguments; trigonometric relationships; **development of axiomatic proof**; and *foundational*/**robust** understanding of non-Euclidean geometries. Well-prepared beginning teachers of mathematics:

a. Recognize and value geometry as a lens to reason with ideas and to solve problems from real world and mathematical domains.

b. Connect geometry and measurement to other mathematical content areas such as: to algebra when they work in the coordinate plane, to ratios and proportional relationships when they explore scaling and scale drawings, to functions when they explore transformations, to number theory when they explore Pythagorean triples, and to calculus when they find the area under a curve.

c. Recognize and describe the levels of geometric understanding (such as van Hiele) as it develops through a progression from investigations to more formal proof and reasoning.

d. Explain, apply, and reason about the Pythagorean theorem (both the statement and the contrapositive), are able to reason through proofs of the theorem based on similarity, area or transformations, and can describe the connections of the Pythagorean theorem to areas of mathematics such as coordinate geometry (distance formula, equations of circles) and trigonometry.

e. Use a variety of tools including dynamic geometry software to investigate and understand variance and invariance of geometric objects and to make and test conjectures.

f. Understand congruence and similarity in terms of transformations and solve problems involving congruence and similarity in multiple ways.

g. Apply the transformation definition of congruence and similarity in terms of rigid motions to establish congruence and similarity criteria and use these criteria to prove theorems about triangles, quadrilaterals, and other geometric figures.

h. Solve problems using transformations, coordinate geometry or Euclidean geometry.

i. Understand the role of definitions, postulates, and axioms and use them to prove theorems about angles, lines, and 2 and 3-dimensional shapes, including the parallel postulate and its connection to Euclidean and **non-Euclidean** geometries.

D.1.7. Essential Concepts in Calculus

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts in calculus, recognizing calculus as the study of change, including instantaneous rates of change and accumulation of values. *Well-prepared beginning middle school (5-9) teachers are expected to have a foundational, basic knowledge of the key concepts in calculus*; the **well-prepared beginning high school (7-12) teachers will have a comprehensive, robust understanding of the topics**. Essential concepts in calculus include limits; rates of change; continuity; fundamental theorem of calculus; and the meaning, techniques, and applications of differentiation and integration. Well-prepared beginning teachers of mathematics:

a. Recognize and value the power of calculus to solve observed phenomenal problems involving change and **accumulation**.

b. Understand how ideas related to algebra, geometry, and functions are involved in understanding and applying calculus concepts.

c. Connect the concepts of limit, derivative, and integration.

d. Use dynamic interactive technology to develop conceptual understanding of key concepts such as derivative or mean value theorem, use graphing technology to analyze functions and their first and second derivatives; and develop notions of limit and understand how sequences and series behave.

e. Apply *foundational*/**robust** definitions, theorems, and concepts from calculus to solve mathematical and contextual problems, attending to the need to verify the hypotheses/conditions for using theorems.

f. Interpret results in the context of a given situation when using calculus to solve a problem.

g. Fluently use and interpret the notation involved in describing and working with limits, derivatives, integrals, **series**, and **sequences**.

h. Connect graphical, algebraic, tabular, and verbal representations of a problem involving rates of change, approximations, or **accumulation**, understanding the advantages and limitations of each.

i. Identify and explain *foundational*/**robust** common underlying structures in concepts involving rate of change, approximations, and **accumulation**.

D.1.8. Essential Concepts in Discrete Mathematics

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematics concepts related to discrete mathematics including the exploration of discrete rather than continuous variables. *Well-prepared beginning middle school (5-9) teachers are expected to have a foundational, basic knowledge of the key concepts in discrete mathematics*; the **well-prepared beginning high school (7-12) teachers will have a comprehensive, robust understanding of the topics**. Essential concepts in discrete mathematics include logic, set theory, counting, number theory, graph theory, **induction,** and recursion. Well-prepared beginning teachers of mathematics:

a. Seek out opportunities to engage in solving problems leveraging strategies and tools from areas of discrete mathematics.

b. Identify ways in which discrete mathematics can connect different mathematical domains.

c. Use a variety of techniques to count and arrange sets of objects (combinatorics), including making the connection of counting to Pascal's triangle.

d. **Apply mathematical induction to prove statements about sequences.**

e. Understand recurrence relations and can reason recursively.

f. Model situations with networks and use graph theory to solve problems.

g. Are familiar with algorithms, their implementation and efficiencies.

h. Identify and analyze common sequences and their characteristics (e.g.,

   Fibonacci, triangular numbers, arithmetic, and geometric progressions).

i. Apply concepts from logic and logical reasoning puzzles and related mathematical problems.

j. Understand and appreciate how discrete mathematics is used in real world situations such as theoretical computer science and cyber security.

D.2. Demonstrate Mathematical Practices

Well-prepared beginning teachers of mathematics demonstrate deep and flexible knowledge of mathematical practices (see Michigan K-12 Standards for Mathematical

Practice). Like their future students, they are able to use these mathematical processes and practices as tools to solve problems and communicate ideas. Well prepared beginning teachers of mathematics:

a. Engage in Mathematical Practices:
    - SMP1: Make sense of problems and persevere in solving them
    - SMP2: Reason abstractly and quantitatively
    - SMP3: Construct viable arguments and critique the reasoning of others
    - SMP4: Model with mathematics
    - SMP5: Use appropriate tools strategically
    - SMP6: Attend to precision
    - SMP7: Look for and make use of structure
    - SMP8: Look for and express regularity in repeated reasoning
b. Understand that doing mathematics is a sense-making activity that calls for perseverance, problem posing, and problem solving.
c. Explain their mathematical thinking using grade-appropriate concepts, procedures, and language, including grade-appropriate definitions and interpretations for key mathematical concepts.
d. Be metacognitive with respect to the practices in their mathematical thinking and highlight these actions and behaviors when they observe them in others.
e. Recognize the interrelationships among the practices and how they support each other, and those that are important to a mathematical investigation.

D.3. Exhibit Productive Mathematical Dispositions

Well-prepared beginning teachers of mathematics expect mathematics to be sensible, useful, and worthwhile for themselves and others, and they believe that all people are capable of thinking mathematically and are able to solve sophisticated mathematical problems with effort. Well-prepared beginning teachers of mathematics:

a. Know that success in mathematics depends on a productive disposition toward the subject.

b. Are reflective practitioners, understanding that student learning of mathematics relies upon them and view their roles as supporting the development of students' robust and powerful mathematical identities.

c. Describe mathematics as a sense-making activity that calls for habits of mind such as curiosity, imagination, inventiveness, risk-taking, and persistence.

d. See sense in mathematics, perceive it as both useful and worthwhile, believe that steady effort in learning mathematics pays off, and see oneself as an effective learner and doer of mathematics.

e. Identify beliefs and classroom practices that produce equitable and inequitable mathematical learning experiences and outcomes for students and seek to create more equitable learning environments.

f. See themselves as perpetual learners of mathematics and look for new and innovative ways to solve problems and seek out new mathematical tools and techniques.

D.4. Analyze the Mathematical Content of Curriculum

Well-prepared beginning teachers of mathematics read, analyze, interpret, and enact standards documents, learning progressions, mathematics curricula, instructional materials, and assessment frameworks for the grades in which they are being prepared to teach. In addition, they should have significant understanding of mathematical standards for adjacent grade levels and a broad knowledge of mathematical standards for all of PK-12. Well-prepared beginning teachers of mathematics:

a. Understand the content within standards documents, learning progressions, mathematics curricula, instructional materials, and assessment frameworks and be able to discuss them with colleagues, administrators, and families of their students in ways that make sense in each audience.

b. Connect standards documents, learning progressions, mathematics curricula, instructional materials, and assessment frameworks and have the commitment to analyze these guides to inform their teaching.

c. Analyze provided instructional resources and formative assessment data to make decisions about the sequencing and time required to teach the content

as well as how to make important connections among the mathematics taught in the grades and/or units before and after what they are teaching.

d. Apply their knowledge of content and practices to critically analyze multiple mathematical instructional resources, to determine whether resources fully address the content and practice expectations described in standards and curriculum documents and promote equitable and effective teaching.

D.5. Use Mathematical Tools and Technology

Well-prepared beginning teachers of mathematics are proficient with tools and technology designed to support mathematical reasoning and sense making, both in doing mathematics themselves and supporting student learning[7] of mathematics.

Well-prepared beginning teachers of mathematics:

a. Select and use tools and technology for solving mathematical problems, for mathematical modeling, and for supporting mathematical reasoning and sense making.

b. Select and use manipulatives (e.g., fold paper, toss coins, demonstrate polyhedra models, measure with protractors) and technology (e.g., interact with an applet displaying slope triangles for a line, find the intersection point of two functions using a graphing calculator, drag a point on a number line, or change values in a spreadsheet to explore the meaning of variable) as a means of developing students' understanding of mathematics.

c. Employ the strategic use of virtual manipulatives and interactive electronic depictions of physical manipulatives, know how these can support sophisticated explorations of mathematical concepts, and make sound decisions about when such tools enhance teaching and learning.

d. Understand the benefits of using physical and virtual manipulatives and make strategic choices between them, recognizing their potential and limitations for students' mathematics learning.

e. Recognize the fast rate at which technologies emerge and are replaced and are committed to staying abreast of new tools.

[7] supporting student learning includes enabling students to do mathematics with the technology and and to use the technology to develop understanding of key mathematical concepts

Glossary

Asset-based Perspective: view of students that foregrounds the useful resources they bring to the learning of mathematics

Conceptual Understanding: comprehension of mathematical concepts, operations, and relations (National Research Council, 2001)

Deficit-based View: view of students that foregrounds what they are lacking when learning mathematics

Equity-Based Pedagogy: approach to teaching that emphasizes rich and rigorous mathematics; elicits and builds on students' mathematical thinking; connects to students' cultural/linguistic knowledge and backgrounds as well as individual interests; facilitates mathematical discourse; minimizes status issues by expanding broader participation and engagement where varied mathematical strengths are valued; and promotes positive mathematical identity and agency (AMTE, 2015)

Explicit Bias: attitudes or stereotypes that affect our understanding, actions, and decisions of which we are conscious

Implicit Bias: attitudes or stereotypes that affect our understanding, actions, and decisions of which we are unconscious (Kirwan Institute, 2018)

Learning progressions: research-based paths describing stages students go through as they move towards mastering a specific mathematical topic or process

Mathematics-Specific Tools: implements used to do or learn mathematics, such as protractors

Multiple Entry Points: various ways for a student to begin engaging with a task

Power and Privilege: ability to make decisions that affect others' opportunities and future outcomes (power) and give preferential treatment (privilege)

Procedural Fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (National Research Council, 2001)

Rigorous Mathematics: mathematics that: 1) calls for strategic and extended thinking, planning, and complex reasoning; 2) takes synthesis and transference of knowledge from one domain to solve a problem in another domain (Aungst, 2014)

Students' Mathematical Identities: dispositions and beliefs students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways in their lives (Aguirre et al., 2013)

Student Agency: expression of a student's identity; level of agency determined by ways in which a student participates in mathematics and expresses their mathematical identity (NCTM, 2018)

Systemic Structures: characteristics of a system that affect the general behavior of the entire system

Appendices

to the Standards for the Preparation of Teachers of Middle Grades (grades 5-9) and High School (grades 7-12) Mathematics

Appendix A: Background and Development of the Standards

Background

Since 2015, the Michigan Department of Education (MDE), in collaboration with Michigan's stakeholders, have been working to revise Michigan's teacher certification structure and improve the preparation of the educator workforce in Michigan. This is in direct alignment with the MDE's Strategic Plan for becoming a Top Ten Education State within Ten Years.

This collaboration has led to the design of a structure that places students at the heart of the system. A key goal of this structure is deeper preparation of teachers to meet the unique learning, developmental, and social-emotional needs of children at each grade level. This structure includes focused grade bands to provide new teachers with specialized knowledge about the students and content they will teach and defined clinical experiences and foundational coursework for each grade band.

Building on the work of the certification restructuring and the revision and adoption of Early Elementary (PK-3) and Upper Elementary (3-6) Standards for Teacher Preparation, stakeholder committees consisting of individuals with expertise in secondary level teacher professional knowledge and skills, English language arts (ELA), and mathematics were convened to revise standards to align with the new middle grades (5-9) and high school (7-12) grade bands. These three areas were selected due to the primacy of importance to teacher preparation programs, the critical need subjects represented, and the alignment with MDE's Whole Child and Literacy foci.

In line with these initiatives, and in response to feedback from educators around the state, each of these sets of standards has a deep focus on equity, shifting the vision of a well-prepared beginning teacher at the secondary level from an emphasis on decontextualized content knowledge and toward an emphasis on classroom practices that address the diverse social, emotional, developmental, and learning needs of the whole child. These standards establish a vision of the middle school and high school classroom as an inclusive space, in which the individual, multifaceted identities of all children are recognized as assets to support their learning and development. The standards do not define the English language arts or mathematics curriculum for these grade bands, but rather define the knowledge, skills, and dispositions teachers must have in order to achieve this vision of an inclusive space that fulfills all children's needs across the several interacting dimensions of the Whole Child.

While the preparation standards for middle grades and high school presented here are in most areas identical for both grade bands, they are not intended to be addressed in isolation from other core elements of the teacher preparation curriculum. Candidates seeking endorsement in either or both grade bands must engage in clinical experiences across the range of grades incorporated in these bands (in accordance with Michigan's Clinical Experiences Requirements) that allow them to demonstrate proficiency in these standards as well as the Core Teaching Practices in authentic classroom contexts.

Expectations for K-12 student performance in these discipline areas also vary across the two grade bands, requiring that enactment of the core instructional competencies and dispositions detailed in these standards may vary in accordance with the K-12 academic standards and locally adopted curriculum being addressed in a specific classroom context.

Development of the Mathematics Teacher Preparation Standards

As a direct result of the restructuring of teacher certification and concurrently with the Professional Knowledge and Skills and English Language Arts standards workgroups, a stakeholder committee representing grades 5-12 mathematics educators and administrators, college and university teacher educators and mathematics educators, and education researchers began meeting in October 2018 to revise secondary mathematics teacher preparation standards to support the new middle grades and high school grade bands.

The stakeholder committee began its work by reviewing Michigan's Standards for the Preparation of Teachers of Mathematics – Secondary (EX), adopted by the State Board of Education (SBE) in 2000, to determine whether they provided adequate guidance to prepare teachers to support students in the 5-9 and 7-12 grade bands in achieving the Michigan K-12 Standards for Mathematics, adopted by the SBE in 2010. The committee considered the question of whether to reaffirm existing Michigan teacher preparation standards, compose new standards, or adopt a national set of standards as Michigan's standards. The stakeholder committee was unanimous in recommending that new standards be composed for Michigan teacher preparation in mathematics.
The committee began by creating a framework for organizing the mathematics standards. This framework allowed for the revision of the standards in such a way that they aligned with current research in the field from such groups as the National Council of Teachers of Mathematics (NCTM) and the Association of Mathematics Teacher Educators (AMTE), supported alignment with MDE's Whole Child and Equity focuses, and ensured a balance and integration of pedagogy and content, all critical aspects of preparation of a strong well-prepared education workforce in mathematics. As this framework was developed, in addition to the three key aspects of Pedagogy, Content, and Dispositions (PCD), several crosscutting focuses were also determined to be important.

Starting with the Michigan Standards for Preparation of Teachers of Mathematics - Secondary (2000) and recently adopted Michigan Standards for Preparation of Teachers of Upper Elementary – Mathematics (2018), key source documents were identified, including NCTM's Standards for Mathematics Teacher Preparation (2019 Draft), AMTE's Standards for Preparing Teachers of Mathematics (2017), and the NCTM publications Principles to Actions: Ensuring Mathematical Success for All (2014) and Catalyzing Change in High School Mathematics: Initiating Critical Conversations (2018). After completion of the initial draft, reviewers from the field of mathematics and mathematics education provided feedback. The reviewers included representatives across the spectrum of PK-12 and teacher preparation programs. This

feedback was compiled and responded to by the committee, in such a way that incorporated all of it into the current draft. Each piece of feedback was discussed and either accepted as is, accepted with modifications, or rejected with rationale.

Public Comment

Following the presentation of the proposed Standards for the Preparation of Teachers in Professional Knowledge and Skills, English Language Arts, and Mathematics in Middle Grades (5-9) and High School (7-12) at the February 11, 2020 State Board of Education meeting, the Office of Educator Excellence solicited public comment on the proposed standards through April 29, 2020. A total of 201 individuals (PK-12 teachers, teacher candidates, school administrators, teacher educators, education organization representatives, parents, and interested citizens) participated in the public comment survey. Feedback was strongly positive, with 78% of respondents agreeing that the proposed standards would improve the preparation of middle grades (5-9) teachers, and 76% agreeing that the proposed standards would improve the preparation of high school (7-12) teachers.

Statements of support for the standards constituted the largest number of open ended comments received for each set of standards in both grade levels. Eighteen respondents specifically praised the standards' inclusive vision of teaching practices to support the needs of individual students in Michigan's increasingly diverse classrooms.

Eleven respondents stated that the standards needed greater emphasis on knowledge and skills related to supporting English learners across content areas in the general education classroom. Within the Professional Knowledge and Skills standards for both grade bands, multiple standards, such as P.1.j and P.1.k, explicitly call attention to the needs of English learners. Other standards, such as P.1.d, are intended to address multiple aspects of student diversity, including English learners. Within the English Language Arts standards, implicit references to English learners in standards ELA.2.b, ELA.2.c, ELA.2.d, ELA.4.b, and ELA.4.c were made explicit to emphasize the need for an intentional focus on preparing teacher candidates to support this population. The mathematics standards frequently reference "all learners" which is intended to be inclusive of English learners therefore changes were not made to these standards. Teacher preparation programs are advised to increase the curriculum's emphasis in this area. Technical assistance resources will be provided to further emphasize the inclusivity of "all learners."

Eleven respondents expressed general support for the standards but noted that their effectiveness will depend on how educator preparations implement them through coursework and clinical experiences. Michigan's Clinical Experiences Requirements provide a detailed framework for clinically-based teacher preparation to ensure that candidates have extensive opportunities to engage with students representing the diversity of Michigan's population in authentic school settings. These experiences are

required to be distributed across the arc of a teacher preparation program, not solely isolated to student teaching. In addition to the Clinical Experiences Requirements, these standards should also be considered alongside the Core Teaching Practices, which together with the Professional Knowledge and Skills standards form the basis for the professional education coursework all teacher candidates will experience, regardless of disciplinary specialization. Supplemental material to be developed during the implementation of these standards will include a glossary and recommended resources to support teacher educators' and teacher candidates' understandings of the standards.

Four respondents expressed concern for the balance between content knowledge and pedagogical training in the standards. All the English language arts and mathematics standards were written in such a way as to intersect the content knowledge and pedagogy needed to effectively deliver instruction. Institutions are expected to provide coursework to ensure beginning teachers have the content knowledge necessary to address the specific skills described in each of the standards. The Professional Knowledge and Skills standards address the teaching skills necessary to build relationships and manage classrooms.

Three respondents expressed concerns with the amount of emphasis the standards place on equity and advocacy. One respondent felt the emphasis was not strong enough, while two respondents argued that it was an unnecessary and harmful overreach in defining teachers' knowledge and skills. At its March 12, 2019 meeting, the Michigan State Board of Education adopted a Resolution on Dignity in Schools, which "reaffirms its commitment to upholding all federal and state civil rights laws; helping school districts enact policies that improve school climate, safety, and do not have a disparate impact on students based on their race, sex, LGBTQ status, disability status, or age; and eliminating the use of ineffective exclusionary discipline and increasing equity for all students." State board approved standards provide ways for the enactment of SBE resolutions supporting equity in schools at the state level. Teachers, as enactors of SBE policy, engage in advocacy for students at the classroom and school level in order to encourage and enact equity for all students in the state.

Furthermore, through its strategic priority of the Whole Child, "MDE believes caring for, supporting, and educating the whole child is an essential part of promoting academic achievement and excellence throughout the Prenatal Through Age 20 (P20) system." These standards are intended to support achievement of this priority by preparing teachers to meet all the needs of the whole child for each individual learner.

Twelve respondents expressed concerns not about the standards, but rather about teacher shortages and the effect narrower grade bands will have on the ability of small schools and districts to make appropriate staffing decisions. These concerns are addressed by the flexibilities and communications included as part of the OEE's proposed appropriate placement policies, which will accompany the implementation of these preparation standards.

Appendix B: General Implementation Resources

Timeline for implementation

- Certification Structure (michigan.gov)

Timeline of Implementation Milestones by Grade Band

Grade Band	Preparation Program Applications Due	Program Review Complete	Expected First Enrollment in New Programs	New MTTCs Offered	Expected First Candidates Graduate	Final Offering of Old MTTC
Early Childhood General and Special Education (B-K)	Nov 30, 2020, Apr 30, 2021, or Nov 30, 2021	Mar 2021, Sep 2021, or Mar 2022	Fall 2021	2022	Spring 2023	Late 2026 (MTTC #106)
Lower Elementary (PreK-3)	Nov 30, 2020, Apr 30, 2021, or Nov 30, 2021	Mar 2021, Sep 2021, or Mar 2022	Fall 2021	Spring 2022	Spring 2023	July 2026 (MTTC #103)
Upper Elementary (3-6)	Nov 30, 2020, Apr 30, 2021, or Nov 30, 2021	Mar 2021, Sep 2021, or Mar 2022	Fall 2021	Spring 2022	Spring 2023	July 2026 (MTTC #103)
Middle Level (5-9) ELA, Math, Professional	Nov 30, 2021, Apr 30, 2022, or Nov 30, 2022	Mar 2022, Sep 2022, or Mar 2023	Fall 2023	2023	Spring 2025	2029 (MTTC #085, #090 #002, #089, #022)
High School (7-12) ELA, Math, Professional	Nov 30, 2021, Apr 30, 2022, or Nov 30, 2022	Mar 2022, Sep 2022, or Mar 2023	Fall 2023	2023	Spring 2025	2029 (MTTC, #085, #090 #002, #089, #022)

General Resources

- Introduction to Michigan Middle Grades and High School Teacher Preparation Standards for Mathematics - YouTube
    - Recorded MDE webinar introduction of the new teacher preparation standards for grades 5-9 and grades 7-12 mathematics teachers in Michigan. Intended to be guidance for educator preparation programs as they embark upon revising programs to meet the new standards. Four of the standards authors describe the key shifts in the new standards.
- AMTE Website - Resources for implementation (requires membership)
    - The AMTE Standards for Preparing Teachers of Mathematics (SPTM) elaborate what beginning teachers of mathematics must know and be able to do as well as the dispositions they must have to increase equity, access, and opportunities for the mathematical success of each of their future students.
    - See also: Call to Action: Expanding Mathematical Knowledge for Teaching in Early College Mathematics
- NCTM Principles to Actions - description
    - NCTM's landmark publication Principles to Actions connects research with practice. Specific, research-based teaching practices that are essential for a high-quality mathematics education for each and every student are combined with core principles to build a successful mathematics program at all levels. Principles to Actions offers guidance to teachers, mathematics coaches, administrators, parents, and policymakers. It includes:
        - Eight research-based essential Mathematics Teaching Practices
        - Conditions, structures, and policies necessary to support the Effective Teaching Practices
        - Implementation strategies for the Common Core State Standards for Mathematics built from Principles and Standards for School Mathematics and designed to attain much higher levels of mathematics achievement for each and every student
        - Unproductive and productive beliefs, obstacles, and key actions that must be understood, acknowledged, and addressed by all stakeholders
        - Strategies for teachers to engage students in mathematical thinking, reasoning, and sense making to significantly strengthen teaching and learning
- NCTM Principles to Actions toolkit
    - These grade-band specific professional learning modules are focused on the Effective Teaching Practices and Guiding Principles from *Principles to Actions: Ensuring Mathematical Success for All*.
- NCTM CAEP materials - NCTM Standards for Teacher Preparation Programs
    - NCTM's Mathematics teacher preparation program recognition standards and review process are used by institutions seeking accreditation from the Council for the Accreditation of Educator Preparation (CAEP). The NCTM

Standards serve as the basis for programs to determine which required assessments provide evidence of candidate mastery of the standards. See also NCTM articles:
    - Justification as an Equity Practice See how to give students equitable access to mathematics, develop their agency, and support their engagement.
    - Tiering Instruction for Seventh-Grade Students A study based on a proportional reasoning unit shows how differentiating for students' ways of thinking can effectively meet diverse learning needs
    - Using Models to Build Fraction Understanding Ponder how a sequence of representations can help create an evolving comprehension of unit, partitioning, and order.
- Statistical Education of Teachers Report from the American Statistical Association
    - This report, The Statistical Education of Teachers ( ), was commissioned by the American Statistical Association (ASA) to clarify MET II's recommendations, emphasizing features of teachers' statistical preparation that are distinct from their mathematical preparation. SET calls for collaboration among mathematicians, statisticians, mathematics educators, and statistics educators to prepare teachers to teach the intellectually demanding statistics in the PreK–12 curriculum, and it serves as a resource to aid those efforts
- Guidelines for Assessment and Instruction in Statistics Education: PreK-12 and College
    - This guide continues to endorse the six recommendations outlined in the original GAISE College Report with some slight revisions. The revised recommendations are: 1. Teach statistical thinking. Teach statistics as an investigative process of problem-solving and decisionmaking. Give students experience with multivariable thinking. 2. Focus on conceptual understanding. 3. Integrate real data with a context and purpose. 4. Foster active learning. 5. Use technology to explore concepts and analyze data. 6. Use assessments to improve and evaluate student learning.
- NRC Active Learning vs Non-active Learning: Active learning increases student performance in science, engineering, and mathematics
    - The studies analyzed here document that active learning leads to increases in examination performance that would raise average grades by a half a letter, and that failure rates under traditional lecturing increase by 55% over the rates observed under active learning. The analysis supports theory claiming that calls to increase the number of students receiving STEM degrees could be answered, at least in part, by abandoning traditional lecturing in favor of active learning.
- Instructional Practices Guide from Mathematical Association of America (MAA)
    - This Instructional Practices Guide aims to share effective, evidence-based practices instructors can use to facilitate meaningful learning for students of mathematics. With that big picture in mind, this guide is written from the perspective that teaching and learning is a force for social change. Beyond

the confines of individual instructors' classrooms, beyond their decisions about what mathematics to teach and how to teach it, there are societal forces that call upon all mathematics instructors to advocate for increased student access to the discipline of mathematics.

- Essential Practices for Literacy Instruction in the Secondary Mathematics Classroom
    - Deliberate, research-supported strategies to motivate, engage, and support reading, writing, speaking, and listening in mathematics
- Michigan Council of Teaching Mathematics blog

Middle Grades Resources

- Middle Level Education and Early Adolescent Development Resources at Association for Middle Level Educators: Our landmark position paper, The Successful Middle School: This We Believe, is recognized as the best articulation of the middle school concept. Since it was first published in 1982, middle school educators have relied on the text to build high-performing school communities using the 5 essential attributes and 18 Characteristics at the core of successful middle schools. The text has grown from a single document into a comprehensive program of tools and resources that educators, schools, and districts can adopt to help ensure student success. Links here of particular interest to Mathematics educators include:
    - Professional Preparation and Credentialing of Middle Level Teachers
    - Young Adolescent Development Archives
    - Resource Center
    - Ohio's Middle Childhood Licensure Study
    - University-Hub Landing Page
- Middle Grades Content Resources:
    - Catalyzing Change in Middle School Mathematics - NCTM
    - The following links contain the learning progressions for Michigan Mathematics Standards at particular grade levels and topics:
        - Draft 3–5 Progression on Number and Operations—Fractions
        - Draft 6–8 Progression on Statistics and Probability
        - Draft 6–8 Progression on Expressions and Equations
        - Draft 6–8 Progression on The Number System; High School, Number
        - Draft 6–7 Progression on Ratios and Proportional Relationships

High School Resources

- High School Content Resources:
    - Catalyzing Change in High School Mathematics - NCTM
    - The following links contain the learning progressions for Michigan Mathematics Standards at particular grade levels and topics:
        - Draft High School Progression on Statistics and Probability
        - Draft High School Progression on Algebra
        - Draft High School Progression on Functions
        - Draft High School Progression on Modeling
        - Draft 7–HS Progression on Geometry
        - Draft High School Progression on Quantity

Mathematics Education Equity Resources

- Social Justice & Anti-Racist Educator Resources

    - The compilation aims to help colleagues begin, continue, and further their own work to FIRST educate themselves and then bring anti-racist teaching to all grade levels and subject areas.
- TODOS: Mathematics for All provides publications, resources and professional learning
    - TODOS: Mathematics for ALL is a professional organization that advocates for equity and excellence in mathematics education for ALL students - in particular, Latina/o students. We promote Social Justice in Mathematics education and provide high-quality resources to help reach our Mission and Goals.
- Equity Committee | AMTE suggested resources at the bottom
    - Providing teaching and learning resources and recommendations appropriate for mathematics teacher educators positioned anywhere along the 'equity' experience continuum to build their knowledge base in equity. Some AMTE Resources of note:
        - Providing Opportunities for Teacher Candidates to Understand Power, Privilege, and Oppression
- MCTM Equity Resources: A Pathway to Equitable Math Instruction
    - Resources and guidance to support Black, LatinX, and Multilingual students to thrive in grades 6-8. A Pathway to Equitable Math Instruction is an integrated approach to mathematics that centers Black, Latinx, and Multilingual students in grades 6-8, addresses barriers to math equity, and aligns instruction to grade-level priority standards. The Pathway offers guidance and resources for educators to use now as they plan their curriculum, while also offering opportunities for ongoing self-reflection as they seek to develop an anti-racist math practice. The toolkit "strides" serve as multiple on-ramps for educators as they navigate the individual and collective journey from equity to anti-racism.
- Mathematics for Social Justice: Resources for the College Classroom
    - Mathematics for Social Justice offers a collection of resources for mathematics faculty interested in incorporating questions of social justice into their classrooms. The book begins with a series of essays from instructors experienced in integrating social justice themes into their pedagogy; these essays contain political and pedagogical motivations as well as nuts-and-bolts teaching advice. The heart of the book is a collection of fourteen classroom-tested modules featuring ready-to-use activities and investigations for the college mathematics classroom.

Instructional Practices for Equity

- Math Is a Verb: Activities and Lessons from Cultures around the World.
    - For most people, the word mathematics is a noun. But for many people in different cultures, mathematics is not simply something they learn in school but something they do as an intrinsic part of their everyday lives. This book is a guide for teachers who would like to enhance their mathematics

instruction by integrating it with examples and activities from cultures throughout the world. It provides culturally situated examples, each linked to Common Core objectives that show how mathematics can be so much more than a story problem or an exercise in a worksheet with little or no context.

- Edutopia: Culturally Responsive Teaching - Articles & videos
    - This collection of articles gives some nice examples of culturally responsive teaching and leads off with examples in mathematics to lead engaging student inquiries.
- Rethinking mathematics: Teaching social justice by the numbers (2nd Ed.).
    - This unique collection of more than 30 articles shows teachers how to weave social-justice principles throughout the math curriculum, and how to integrate social-justice math into other curricular areas as well. "Rethinking Mathematics" presents teaching ideas, lesson plans and reflections by practicing classroom teachers and distinguished mathematics educators. This is real-world math--math that helps students analyze problems as they gain essential academic skills. "Rethinking Mathematics" will help teachers develop students' understanding of society and prepare them to be critical, active participants in a democracy. This book is divided into four parts. Part I, Viewing Math Broadly, contains the following: (1) Teaching Math Across the Curriculum (Bob Peterson); (2) Reading the World with Math (Marilyn Frankenstein); (3) Race, Retrenchment, and the Reform of School Mathematics (William F. Tate); and (4) Historical, Cultural, and Social Implications of Mathematics (S. E. Anderson). Part II, Infusing Social Justice into Math Classes, contains the following: (5) "Home Buying While Brown or Black" (Eric Gutstein); (6) Sweatshop Accounting (Larry Steele); (7) Chicanos Have Math in Their Blood (Luis Ortiz-Franco); (8) Understanding Large Numbers (Bob Peterson); (9) "With Math, It's Like You Have More Defense" (Erin E. Turner and Beatriz T. Font Strawhun); (10) "Radical Equations": A Review of the Book by Bob Moses and Charles E. Cobb Jr. (David Levine); (11) The Geometry of Inequality (Andrew Brantlinger); (12) Integrals and Equity (Megan Staples); (13) Math, Maps, and Misrepresentation (Eric Gutstein); and (14) Multicultural Math (Claudia Zaslavsky). Part III, Infusing Social Justice Math into Other Curricular Areas, contains the following: (15) Ten Chairs of Inequality (Polly Kellogg); (16) Write the Truth: Presidents and Slaves (Bob Peterson); and (17) The Transnational Capital Auction (Bill Bigelow). Part IV, Resources for Rethinking, contains the following: (18) Websites, Math Curriculum and Pedagogy, Books with Theoretical/Academic Perspectives, Curriculum Guides/Resources, Children's Books, Sources for Maps, and More. An index is included.
- Powerful Practices for Supporting English Learners: Elevating Diverse Assets and Identities
    - This practical guide, grounded in compelling research and organized around essential questions and answers, is designed to help all educators build on

their current competencies to authentically harmonize home languages and cultures in the classroom.

Instructional Mindsets for Equity

- Culturally Responsive Teaching and The Brain
    - The thesis of this research is using culturally responsive mathematics teaching as a way to decrease the achievement gap. Culturally responsive pedagogy has shown great promise in meeting this need, but many educators still struggle with its implementation. In this book, Zaretta Hammond draws on cutting-edge neuroscience research to offer an innovative approach for designing and implementing brain-compatible culturally responsive instruction.
- Social and Political Dimensions of Mathematics Education
    - This book examines the current thinking on five critical social and political areas in mathematics education. It focuses on material conditions in teaching and learning, and details features of social life and their influence on mathematics teaching, learning and achievement. Following an introduction, the first section addresses equitable access and participation in quality mathematics education. It explores this issue in different contexts and from different ideological perspectives. The second section traces the emergence and development of the notion of activism in mathematics education in theory, in the literature, in research and in practice. The third section then moves on to explore current research on the political forces at work in identity, subjectivity and (dis)ability within mathematics education, showing how emphasis on language and discourse provides information for this research, and how new directions are being pursued to address the diverse material conditions that shape learning experiences in mathematics education. Economic factors behind mathematics achievement form the topic of section four, which examines the political dimensions of mathematics education through the influence of national and global economic structures. The final section addresses distribution of power and cultural regimes of truth, based on the premise that although often deemed apolitical, mathematics and mathematics education are highly political institutions in our society. The book concludes with a summary and recommendations for the future.
- The Complexities of Culturally Relevant Pedagogy: A Case Study of Two Secondary Mathematics Teachers and Their ESOL Students.
    - Culturally relevant pedagogy is not well understood as an instructional strategy in the mathematics classroom. This study reveals the challenges two teachers faced when they implemented a pilot project with ninth and tenth grade ESOL students.
- Mathematics for Human Flourishing
    - For mathematician Francis Su, a society without mathematical affection is like a city without concerts, parks, or museums. To miss out on

mathematics is to live without experiencing some of humanity's most beautiful ideas. In this profound book, written for a wide audience but especially for those disenchanted by their past experiences, an award-winning mathematician and educator weaves parables, puzzles, and personal reflections to show how mathematics meets basic human desires—such as for play, beauty, freedom, justice, and love—and cultivates virtues essential for human flourishing.

Mathematics Education Technology Resources

- Digital Technology in Mathematics Education: Why It Works (Or Doesn't) in Selected Regular Lectures from the 12th International Congress on Mathematical Education
    - The integration of digital technology confronts teachers, educators and researchers with many questions. What is the potential of ICT for learning and teaching, and which factors are decisive in making it work in the mathematics classroom? To investigate these questions, six cases from leading studies in the field are described, and decisive success factors are identified. This leads to the conclusion that crucial factors for the success of digital technology in mathematics education include the design of the digital tool and corresponding tasks exploiting the tool's pedagogical potential, the role of the teacher and the educational context.
- Preparing to Teach Mathematics with Technology
    - Since 2005, Preparing to Teach Mathematics with Technology (PTMT) project has developed research-based materials for mathematics teacher education focused on using technology tools for teaching major topics in middle and secondary mathematics in Algebra, Statistics, and Geometry, all available for free.
- Tech Talk
    - Tech Talk is an informal blog for AMTE members, curated by our Technology Committee to support mathematics teacher education.

Appendix C: Grade Band Comparison for Content Standards (D.1)

See also Standard D1 for a combined list of content performance expectations for both grade bands

Middle Grades Content Domain: D. Understanding Mathematics Concepts

D.1. Know Relevant Mathematical Content

Well-prepared beginning teachers of mathematics have deep and flexible knowledge of core mathematical concepts and procedures that they will teach, along with knowledge both beyond what they will teach and foundational to those core concepts and procedures.

D.1.1. Essential Concepts in Number for Middle Grades Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts in number including flexibly applying computational procedures, using real and rational numbers in contexts, solving problems involving numbers and evaluating the solutions. Essential concepts in number for middle grades (5-9) teachers include integers, fractions, number theory, operations (including exponentiation) and their meanings and properties and meanings of complex numbers.

Well Prepared beginning teachers of middle grades mathematics:

a. Demonstrate "number sense"- flexible reasoning with and about whole numbers, integers and rational numbers in a variety of situations and applications through opportunities such as composing and decomposing numbers and number talks.
b. Describe the underlying structure of the real number system and the learning progression for the development of number across the grades from kindergarten through high school.
c. Identify and apply a variety of strategies to compare and estimate rational and irrational numbers.
d. Understand and are fluent in using operations and appropriate notation, including exponentiation, with rational numbers, and can apply and justify multiple strategies for adding, subtracting, multiplying, and dividing rational numbers.
e. Apply and connect concepts such as factor, prime, divisible, and multiple to particular numbers and sets of numbers.
f. Reason about and prove basic theorems about real numbers (e.g., the product of two negative numbers is positive, $\sqrt{2}$ is irrational, or the product of two odd numbers will be odd).
g. Use technology to investigate certain numbers (e.g., value of pi, compare the relative size of two numerical expressions, or evaluate limiting processes).

h. Understand how complex numbers are related to the solutions of quadratic equations.

D.1.2. Essential Concepts in Ratios and Proportional Relationships for Middle Grades Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts related to ratios and proportional relationships including rates and percents and their applications in contexts. Essential concepts in ratios and proportional relationships for middle grades (5-9) teachers include understanding ratios; that equivalent ratios connect to proportional relationships ($y=kx$) and to graphs through the origin (0,0); and that percents are a particular form of a ratio and are invaluable tools in comparing quantities when the group sizes are different.

Well-prepared beginning teachers of middle grades mathematics:

a. Describe the learning progression for the development of proportional reasoning across the grades from kindergarten through high school.
b. Identify situations in which ratios can be a tool to solve problems and apply a variety of strategies such as ratio tables, double number lines, and unit rates to solve problems involving ratios.
c. Understand ratios as paired quantities that vary together in the same relationship, distinct from a single number, recognizing that a ratio a:b may be associated with a value a/b (if b is not 0) and can describe the differences and similarities between ratios and fractions.
d. Identify and use equivalent ratios, reason about the role of multiplication and addition in generating equivalent ratios and recognize that the sum of equivalent ratios is another equivalent ratio.
e. Associate a unit rate with a ratio, recognize that equivalent ratios have the same unit rate, and recognize this unit rate as the constant of proportionality (k) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
f. Connect the constant of proportionality to the slope of a line through the origin and to the equations of such a line $a:b \rightarrow y = (b/a)x$, and describe how this perspective relates to the general concept of linearity.
g. Reason about contextual situations, identifying those that involve proportional relationships and use a variety of strategies for solving problems involving proportions, including finding a unit rate.
h. Connect ratios to scale factors, both within and between scaled figures, and relate scale factors to similar shapes, including how scaling a figure affects the area and volume of the scaled figures.
i. Describe percentage as a particular ratio of a quantity to 100, and apply this understanding to solve a variety of contextual problems.

D.1.3. Essential Concepts in Algebra for Middle Grades Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major concepts in algebra including how algebra can be used systematically to represent patterns and relationships among numbers and other objects, to analyze change, and to model everyday events and problems of life and society. Well-prepared beginning middle school (5-9) teachers will have a comprehensive, robust understanding of the essential concepts related to algebra content at the 5-9 level. Essential concepts in algebra include understanding the role of and generating equivalent expressions, solving equations and inequalities in real numbers, and graphing algebraic relationships.

Well-prepared beginning teachers of middle grades mathematics:

a. Explain how algebra as the language of generalization is useful for describing patterns and relationships.
b. Appreciate the value that new technologies can bring to developing algebraic understandings and strategically employ them in improving learning experiences in algebra for all students.
c. Describe how algebraic concepts build from arithmetic and are connected to other content areas, such as geometry, statistics, and calculus.
d. Describe the role of and be able to apply definitions, reasoning and proof in algebra including identifying conditions under which theorems are valid, recognizing contradiction as a proof strategy, and using conjectures to investigate algebraic relationships.
e. Use different technologies to enhance the learning of mathematics such as computer algebra systems to investigate algebraic structures and to check results; spreadsheets to produce and explore regularity in repeated reasoning; graphs to explore algebraic relationships; and interactive dynamic technologies to develop conceptual understanding of key algebraic topics.
f. Interpret the structure of an algebraic expression in terms of a context and understand that structure can provide insight into a mathematical situation.
g. Connect symbolic, graphical, tabular, and verbal representations of a problem or situation and explain any advantages and disadvantages of each representation for the given problem or situation.
h. Use algebra as a tool to solve contextual problems including identifying variables, formulating an algebraic model, manipulating and analyzing the model, interpreting the results, and validating the conclusions.
i. Explain and justify routine procedures involved in manipulating expressions and solving equations including the use of the properties related to multiplication, addition, and equality.

D.1.4 Essential Concepts in Functions for Middle Grades Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge, and applications of the major concepts related to function, including models of real-world contexts where one variable determines another. Essential concepts in function for middle grades (5-9) teachers include the notion of functions as a mapping[8], the characteristics of different function families (e.g., linear, exponential, polynomial, absolute value, piece-wise defined) and creating and interpreting different representations of functions.

Well-prepared beginning teachers of middle grades mathematics:

a. Recognize the value of function as the language and organizational structure in the analysis of mathematical relationships.
b. Understand how algebra concepts are related to the ideas of function and that not all algebraic equations are functions.
c. Represent functions, with and without technology, in a variety of ways including mapping diagrams, function notation, recursive definitions, tables, and graphs.
d. Connect members of the same function family and identify distinguishing attributes (structure) common to all functions within that family.
e. Compare function families and describe their similarities and differences. (e.g. linear functions are additive, exponential functions are multiplicative).
f. Describe and reason about key features of the graphs of functions, using appropriate language, (e.g. zeros, intercepts, rate of change, increasing/decreasing and maximum/minimum values); associate symbolic representations with these features and interpret them in both mathematical and real-world contexts.
g. Model a wide variety of real situations using functions and understand the processes of making and changing assumptions, assigning variables, and finding solutions to contextual problems.
h. Apply one or more function transformations to a representation (verbal, symbolic, graphical or tabular) of a function and explain the effects and results on other representations.

D.1.5 Essential Concepts in Statistics and Probability for Middle Grades Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major concepts in statistics and probability including understanding the statistical investigation process (formulate questions, collect data or select portions of given data sets, analyze data, interpret results). Essential concepts in statistics and probability for middle grades (5-9) teachers include aspects of data literacy, visualizing and summarizing data

[8] Reworded from original to ensure accuracy, as defining the term was unnecessary: "Essential concepts in function include mapping from one set to another in which each element of the second set is uniquely paired with one element from the first, recognizing . . ."

recognizing the foundational role of variability, statistical inference, and basic ideas related to probability.

Well-prepared beginning teachers of middle grades mathematics:

a. Describe the difference between the way conclusions are made in mathematics (deductive and deterministic) and in statistics (inductive and account for uncertainty) and appreciate that statistical reasoning is always grounded in a context in the presence of variability.
b. Describe statistics as an investigative process of problem-solving and decision making and explain how proficiency in statistical thinking matures as learners gain more knowledge and experience with variability (the developmental levels as outlined in the Guidelines for Assessment and Instruction in Statistics Education).
c. Use real data with a context and purpose, hands on activities and active learning and technology to explore concepts and to manage and analyze data in developing understanding of statistical ideas.
d. Understand and explain measures of center and variability[9] and relate them to distributions of data.
e. Identify the role of randomization and chance in determining the probability of events.
f. Draw inferences about a population based on a sample in light of sampling variability using simulation-based techniques that lead to informal inference procedures.
g. Evaluate reports based on data, reasoning critically and asking questions about the implementation of the statistical investigation process and critique the ways in which numbers and graphical representations are used in the media, mathematical contexts, and everyday discussions.
h. Connect the probability of an outcome to a long-run relative frequency of an outcome and investigate chance processes, developing, using, and evaluating probability models.
i. Analyze and describe the association between variables considering whether causation can be established.
j. Design a basic observational study based on the statistical investigation process.
k. Reflect upon data from implementation of classroom practices to enhance their own craft.

D.1.6 Essential Concepts in Geometry, Trigonometry, and Measurement for Middle Grades Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major

[9] updated from the original "Understand and explain measures of center and spread" for clarity

concepts in geometry, recognizing geometry as a systematic study of size, shape, location and navigation to provide a lens to view and model the world. Essential concepts in geometry, trigonometry, and measurement for middle grades (5-9) teachers include properties of one, two, and three-dimensional shapes; measurement techniques and units; transformations; geometric arguments; trigonometric relationships; and foundational understanding of non-Euclidean geometries.

Well-prepared beginning teachers of middle grades mathematics:

a. Recognize and value geometry as a lens to reason with ideas and to solve problems from real world and mathematical domains.
b. Connect geometry and measurement to other mathematical content areas such as: to algebra when they work in the coordinate plane, to ratios and proportional relationships when they explore scaling and scale drawings, to functions when they explore transformations, to number theory when they explore Pythagorean triples, and to calculus when they find the area under a curve.
c. Recognize and describe the levels of geometric understanding (such as van Hiele) as it develops through a progression from investigations to more formal proof and reasoning.
d. Explain, apply, and reason about the Pythagorean theorem (both the statement and the contrapositive), are able to reason through proofs of the theorem based on similarity, area or transformations, and can describe the connections of the Pythagorean theorem to areas of mathematics such as coordinate geometry (distance formula, equations of circles) and trigonometry.
e. Use a variety of tools including dynamic geometry software to investigate and understand variance and invariance of geometric objects and to make and test conjectures.
f. Understand congruence and similarity in terms of transformations and solve problems involving congruence and similarity in multiple ways.
g. Apply the transformation definition of congruence and similarity in terms of rigid motions to establish congruence and similarity criteria and use these criteria to prove theorems about triangles, quadrilaterals, and other geometric figures.
h. Solve problems using transformations, coordinate geometry or Euclidean geometry.
i. Understand the role of definitions, postulates, and axioms and use them to prove theorems about angles, lines, and 2 and 3-dimensional shapes, including the parallel postulate and its connection to Euclidean geometries.

D.1.7. Essential Concepts in Calculus for Middle Grades Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts in calculus, recognizing calculus as the study of change, including instantaneous rates of change and accumulation of values. Well-prepared

beginning middle school (5-9) teachers are expected to have a foundational, basic knowledge of the essential concepts in calculus. Essential concepts in calculus include limits; rates of change; continuity; fundamental theorem of calculus; and the meaning, techniques, and applications of differentiation and integration.

Well-prepared beginning teachers of middle grades mathematics:

a. Recognize and value the power of calculus to solve observed phenomenal problems involving change.
b. Understand how ideas related to algebra, geometry, and functions are involved in understanding and applying calculus concepts.
c. Connect the concepts of limit, derivative, and integration.
d. Use dynamic interactive technology to develop conceptual understanding of key concepts such as derivative or mean value theorem, use graphing technology to analyze functions and their first and second derivatives; and develop notions of limit and understand how sequences and series behave.
e. Apply foundational definitions, theorems, and concepts from calculus to solve mathematical and contextual problems, attending to the need to verify the hypotheses/conditions for using theorems.
f. Interpret results in the context of a given situation when using calculus to solve a problem.
g. Fluently use and interpret the notation involved in describing and working with limits, derivatives, and integrals.
h. Connect graphical, algebraic, tabular, and verbal representations of a problem involving rates of change, approximations, understanding the advantages and limitations of each.
i. Identify and explain foundational common underlying structures in concepts involving rate of change, approximations.

D.1.8. Essential Concepts in Discrete Mathematics for Middle Grades Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematics concepts related to discrete mathematics including the exploration of discrete rather than continuous variables. Well-prepared beginning middle school (5-9) teachers are expected to have a foundational, basic knowledge of the essential concepts in discrete mathematics. Essential concepts in discrete mathematics include logic, set theory, counting, number theory, graph theory, and recursion.

Well-prepared beginning teachers of middle grades mathematics:

a. Seek out opportunities to engage in solving problems leveraging strategies and tools from areas of discrete mathematics.
b. Identify ways in which discrete mathematics can connect different mathematical domains.

c. Use a variety of techniques to count and arrange sets of objects (combinatorics), including making the connection of counting to Pascal's triangle.
d. Understand recurrence relations and can reason recursively.
e. Model situations with networks and use graph theory to solve problems.
f. Are familiar with algorithms, their implementation and efficiencies.
g. Identify and analyze common sequences and their characteristics (e.g., Fibonacci, triangular numbers, arithmetic, and geometric progressions).
h. Apply concepts from logic and logical reasoning puzzles and related mathematical problems.
i. Understand and appreciate how discrete mathematics is used in real world situations such as theoretical computer science and cyber security.

High School Content Domain: D. Understanding Mathematics Concepts

D.1. Know Relevant Mathematical Content

Well-prepared beginning teachers of mathematics have deep and flexible knowledge of core mathematical concepts and procedures that they will teach, along with knowledge both beyond what they will teach and foundational to those core concepts and procedures.

D.1.1. Essential Concepts in Number for High School Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts in number including flexibly applying computational procedures, using real and rational numbers in contexts, solving problems involving numbers and evaluating the solutions. Essential concepts in number for high school teachers (7-12) include number theory, operations (including exponentiation) and their meanings and properties, meanings of complex numbers, operations with complex numbers, matrices, vectors, and graphing in a polar coordinate system.

Well-prepared beginning teachers of high school mathematics:

a. Demonstrate "number sense"- flexible reasoning with and about whole numbers, integers and rational numbers in a variety of situations and applications through opportunities such as composing and decomposing numbers and number talks.
b. Describe the underlying structure of the real and complex number system and the learning progression for the development of number across the grades from kindergarten through high school.
c. Identify and apply a variety of strategies to compare and estimate rational and irrational numbers.
d. Understand and are fluent in using operations and appropriate notation, including exponentiation and logarithms, with complex numbers, and can apply and justify multiple strategies for adding, subtracting, multiplying, and dividing complex numbers.
e. Apply and connect concepts such as factor, prime, divisible, and multiple to particular numbers and sets of numbers.
f. Reason about and prove basic theorems about real numbers (e.g., the product of two negative numbers is positive, $\sqrt{2}$ is irrational, or the product of two odd numbers will be odd).
g. Use technology to investigate certain numbers (e.g., value of pi, compare the relative size of two numerical expressions, evaluate limiting processes, or compute with matrices).
h. Understand how complex numbers are related to the solutions of polynomial equations.

i. Recognize that vectors and polar coordinates are useful mathematical tools to describe both location and direction and are able to employ these tools in contextual situations.
j. Recognize that matrices are a shorthand notation for organizing information and for carrying out computations on that information.

D.1.2. Essential Concepts in Ratios and Proportional Relationships for High School Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts related to ratios and proportional relationships including rates and percents and their applications in contexts. Essential concepts in ratios and proportional relationships for high school teachers (7-12) include understanding ratios; that equivalent ratios connect to proportional relationships (y=kx) and to graphs through the origin (0,0); and that percents are a particular form of a ratio and are invaluable tools in comparing quantities when the group sizes are different.

Well-prepared beginning teachers of high school mathematics:

a. Describe the learning progression for the development of proportional reasoning across the grades from kindergarten through high school.
b. Identify situations in which ratios can be a tool to solve problems and apply a variety of strategies such as ratio tables, double number lines, and unit rates to solve problems involving ratios.
c. Understand ratios as paired quantities that vary together in the same relationship, distinct from a single number, recognizing that a ratio a:b may be associated with a value a/b (if b is not 0) and can describe the differences and similarities between ratios and fractions.
d. Identify and use equivalent ratios, reason about the role of multiplication and addition in generating equivalent ratios and recognize that the sum of equivalent ratios is another equivalent ratio.
e. Associate a unit rate with a ratio, recognize that equivalent ratios have the same unit rate, and recognize this unit rate as the constant of proportionality (k) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
f. Connect the constant of proportionality to the slope of a line through the origin and to the equations of such a line $a:b \rightarrow y= (b/a)x$, and describe how this perspective relates to the general concept of linearity.
g. Reason about contextual situations, identifying those that involve proportional relationships and use a variety of strategies for solving problems involving proportions, including finding a unit rate.

h. Connect ratios to scale factors, both within and between scaled figures, and relate scale factors to similar shapes, including how scaling a figure affects the area and volume of the scaled figures.
i. Describe percentage as a particular ratio of a quantity to 100, and apply this understanding to solve a variety of contextual problems.

D.1.3. Essential Concepts in Algebra for High School Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major concepts in algebra including how algebra can be used systematically to represent patterns and relationships among numbers and other objects, to analyze change, and to model everyday events and problems of life and society. The well-prepared beginning high school (7-12) teacher will have a comprehensive, robust, understanding of algebra and a basic understanding through linear algebra. Essential concepts in algebra include understanding the role of and generating equivalent expressions, solving equations and inequalities in both real and complex numbers, and graphing algebraic relationships. Additional concepts should include algebra from a more theoretical approach such as relationships among groups, rings, and fields and concepts from linear algebra.

Well-prepared beginning teachers of high school mathematics:

a. Explain how algebra as the language of generalization is useful for describing patterns and relationships.
b. Appreciate the value that new technologies can bring to developing algebraic understandings and strategically employ them in improving learning experiences in algebra for all students.
c. Describe how algebraic concepts build from arithmetic and are connected to other content areas, such as geometry, statistics, and calculus.
d. Describe the role of and be able to apply definitions, reasoning and proof in algebra including identifying conditions under which theorems are valid, recognizing contradiction as a proof strategy, and using conjectures to investigate algebraic relationships.
e. Use different technologies to enhance the learning of mathematics such as computer algebra systems to investigate algebraic structures and to check results; spreadsheets to produce and explore regularity in repeated reasoning; graphs to explore algebraic relationships; and interactive dynamic technologies to develop conceptual understanding of key algebraic topics.
f. Interpret the structure of an algebraic expression in terms of a context and understand that structure can provide insight into a mathematical situation.
g. Connect symbolic, graphical, tabular, and verbal representations of a problem or situation and explain any advantages and disadvantages of each representation for the given problem or situation.
h. Use algebra as a tool to solve contextual problems including identifying variables, formulating an algebraic model, manipulating and analyzing the model, interpreting the results, and validating the conclusions.

i. Explain and justify routine procedures involved in manipulating expressions and solving equations including the use of the properties related to multiplication, addition, and equality.

D.1.4 Essential Concepts in Functions for High School Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge, and applications of the major concepts related to function, including models of real-world contexts where one variable determines another. The differentiation between a well-prepared middle grades (5-9) and high school (7-12) teacher lies in their additional level of expertise in the various types of functions noted below. Essential concepts in function include the notion of functions as a mapping[10], the characteristics of different function families (e.g., linear, exponential, polynomial, absolute value, piece-wise defined, trigonometric, logarithmic) and creating and interpreting different representations of functions.

Well-prepared beginning teachers of high school mathematics:

a. Recognize the value of function as the language and organizational structure in the analysis of mathematical relationships.
b. Understand how algebra concepts are related to the ideas of function and that not all algebraic equations are functions.
c. Represent functions, with and without technology, in a variety of ways including mapping diagrams, function notation, recursive definitions, tables, and graphs.
d. Connect members of the same function family and identify distinguishing attributes (structure) common to all functions within that family.
e. Compare function families and describe their similarities and differences. (e.g. linear functions are additive, exponential functions are multiplicative).
f. Describe and reason about key features of the graphs of functions, using appropriate language, (e.g. zeros, intercepts, rate of change, increasing/decreasing and maximum/minimum values; asymptotes); associate symbolic representations with these features and interpret them in both mathematical and real-world contexts.
g. Model a wide variety of real situations using functions and understand the processes of making and changing assumptions, assigning variables, and finding solutions to contextual problems.

[10] Reworded from original to ensure accuracy, as defining the term was unnecessary: "Essential concepts in function include mapping from one set to another in which each element of the second set is uniquely paired with one element from the first, recognizing . . ."

h. Apply one or more function transformations to a representation (verbal, symbolic, graphical or tabular) of a function and explain the effects and results on other representations.

D.1.5 Essential Concepts in Statistics and Probability for High School Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major concepts in statistics and probability including understanding the statistical investigation process (formulate questions, collect data or select portions of given data sets, analyze data, interpret results). Essential concepts in statistics and probability for high school (7-12) teachers include aspects of data literacy, visualizing and summarizing data recognizing the foundational role of variability, statistical inference, and basic ideas related to probability.

Well-prepared beginning teachers of high school mathematics:

a. Describe the difference between the way conclusions are made in mathematics (deductive and deterministic) and in statistics (inductive and account for uncertainty) and appreciate that statistical reasoning is always grounded in a context in the presence of variability.
b. Describe statistics as an investigative process of problem-solving and decision making and explain how proficiency in statistical thinking matures as learners gain more knowledge and experience with variability (the developmental levels as outlined in the Guidelines for Assessment and Instruction in Statistics Education).
c. Use real data with a context and purpose, hands on activities and active learning and technology to explore concepts and to manage and analyze data in developing understanding of statistical ideas.
d. Understand and explain measures of center and variability[11] and relate them to distributions of data.
e. Identify the role of randomization and chance in determining the probability of events and how randomization is connected to the scope of inference.
f. Draw inferences about a population based on a sample in light of sampling variability using simulation-based techniques that lead to formal inference procedures.
g. Evaluate reports based on data, reasoning critically and asking questions about the implementation of the statistical investigation process and critique the ways in which numbers and graphical representations are used in the media, mathematical contexts, and everyday discussions.
h. Describe the differences among types of studies (survey, experiment, and observational) and number and type of variables (quantitative and categorical) determine the scope and validity of statistical inferences.

[11] updated from the original "Understand and explain measures of center and spread" for clarity

i. Connect the probability of an outcome to a long-run relative frequency of an outcome and investigate chance processes, developing, using, and evaluating probability models.
j. Analyze and describe the association between variables including experience with multivariate thinking and considering whether causation can be established.
k. Design a basic observational or experimental study based on the statistical investigation process.
l. Reflect upon data from implementation of classroom practices to enhance their own craft.

D.1.6 Essential Concepts in Geometry, Trigonometry, and Measurement for High School Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major concepts in geometry, recognizing geometry as a systematic study of size, shape, location and navigation to provide a lens to view and model the world. Essential concepts in geometry, trigonometry, and measurement for high school (7-12) teachers include properties of one, two, and three-dimensional shapes; measurement techniques and units; transformations; geometric arguments; trigonometric relationships; development of axiomatic proof; and robust understanding of non-Euclidean geometries.

Well-prepared beginning teachers of high school mathematics:

a. Recognize and value geometry as a lens to reason with ideas and to solve problems from real world and mathematical domains.
b. Connect geometry and measurement to other mathematical content areas such as: to algebra when they work in the coordinate plane, to ratios and proportional relationships when they explore scaling and scale drawings, to functions when they explore transformations, to number theory when they explore Pythagorean triples, and to calculus when they find the area under a curve.
c. Recognize and describe the levels of geometric understanding (such as van Hiele) as it develops through a progression from investigations to more formal proof and reasoning.
d. Explain, apply, and reason about the Pythagorean theorem (both the statement and the contrapositive), are able to reason through proofs of the theorem based on similarity, area or transformations, and can describe the connections of the Pythagorean theorem to areas of mathematics such as coordinate geometry (distance formula, equations of circles) and trigonometry.
e. Use a variety of tools including dynamic geometry software to investigate and understand variance and invariance of geometric objects and to make and test conjectures.
f. Understand congruence and similarity in terms of transformations and solve problems involving congruence and similarity in multiple ways.

g. Apply the transformation definition of congruence and similarity in terms of rigid motions to establish congruence and similarity criteria and use these criteria to prove theorems about triangles, quadrilaterals, and other geometric figures.
h. Solve problems using transformations, coordinate geometry or Euclidean geometry.
i. Understand the role of definitions, postulates, and axioms and use them to prove theorems about angles, lines, and 2 and 3-dimensional shapes, including the parallel postulate and its connection to Euclidean and non-Euclidean geometries.

D.1.7. Essential Concepts in Calculus for High School Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematical concepts in calculus, recognizing calculus as the study of change, including instantaneous rates of change and accumulation of values. Well-prepared beginning high school (7-12) teachers will have a comprehensive, robust understanding of the topics. Essential concepts in calculus include limits; rates of change; continuity; fundamental theorem of calculus; and the meaning, techniques, and applications of differentiation and integration.

Well-prepared beginning teachers of high school mathematics:

a. Recognize and value the power of calculus to solve observed phenomenal problems involving change and accumulation.
b. Understand how ideas related to algebra, geometry, and functions are involved in understanding and applying calculus concepts.
c. Connect the concepts of limit, derivative, and integration.
d. Use dynamic interactive technology to develop conceptual understanding of key concepts such as derivative or mean value theorem, use graphing technology to analyze functions and their first and second derivatives; and develop notions of limit and understand how sequences and series behave.
e. Apply robust definitions, theorems, and concepts from calculus to solve mathematical and contextual problems, attending to the need to verify the hypotheses/conditions for using theorems.
f. Interpret results in the context of a given situation when using calculus to solve a problem.
g. Fluently use and interpret the notation involved in describing and working with limits, derivatives, integrals, series, and sequences.
h. Connect graphical, algebraic, tabular, and verbal representations of a problem involving rates of change, approximations, or accumulation, understanding the advantages and limitations of each.
i. Identify and explain robust common underlying structures in concepts involving rate of change, approximations, and accumulation.

D.1.8. Essential Concepts in Discrete Mathematics for High School Teachers

Well-prepared beginning teachers of mathematics demonstrate and apply conceptual understanding, procedural fluency, factual knowledge and applications of the major mathematics concepts related to discrete mathematics including the exploration of discrete rather than continuous variables. Well-prepared beginning high school (7-12) teachers will have a comprehensive, robust, understanding of the essential concepts. Essential concepts in discrete mathematics include logic, set theory, counting, number theory, graph theory, induction, and recursion.

Well-prepared beginning teachers of high school mathematics:

a. Seek out opportunities to engage in solving problems leveraging strategies and tools from areas of discrete mathematics.
b. Identify ways in which discrete mathematics can connect different mathematical domains.
c. Use a variety of techniques to count and arrange sets of objects (combinatorics), including making the connection of counting to Pascal's triangle.
d. Apply mathematical induction to prove statements about sequences.
e. Understand recurrence relations and can reason recursively.
f. Model situations with networks and use graph theory to solve problems.
g. Are familiar with algorithms, their implementation and efficiencies.
h. Identify and analyze common sequences and their characteristics (e.g., Fibonacci, triangular numbers, arithmetic, and geometric progressions).
i. Apply concepts from logic and logical reasoning puzzles and related mathematical problems.
j. Understand and appreciate how discrete mathematics is used in real world situations such as theoretical computer science and cyber security.

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